



RESEARCH PAPER

Fractional-order brucellosis transmission model between interspecies with a saturated incidence rate

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Abstract

In this study, brucellosis dynamics between interspecies are discussed with the Atangana-Baleanu fractional derivative to examine the transmission of brucellosis by its behavior. The recovered compartment, recruitment, and natural death rate for humans are considered for the fractional order model to analyze the transmission dynamics in more detail from an epidemiological point of view. Additionally, the saturated incidence rate is suggested for brucellosis as indirectly transmitted to individuals from the environment. By fixed point theory, it is verified that developed fractional transmission dynamics have a unique solution. The model under consideration employs the Adams-type predictor-corrector method for numerical solution. All comparative results are plotted by MATLAB.

Keywords: Atangana-Baleanu derivative; brucellosis; existence and uniqueness; fixed point theory; mathematical modeling

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1 Introduction

Brucellosis is a zoonotic illness that can be transferred to individuals by direct contact with infected animals or indirectly from *Brucella* in a contaminated environment [1]. Humans are transmitted by it in various manners. Firstly, humans can contract brucellosis by consuming rare cooked meat or unpasteurized dairy products. Second, researchers studying bacteria in a laboratory environment can become infected by breathing in *Brucella*. Finally, it can be transmitted to veterinarians or staff working who come into intimate touch with the skin deformation or droppings of infected animals. Furthermore, it is known that human-to-human transmission is extremely uncommon [2]. Even though the mortality from the disease is negligible in humans, brucellosis causes serious organ damage and can continue for several years. Apart from that, the case is not thought to be so

severe in animals. However, the infection can lead to significant financial damage by decreasing infant survival, milk production, reproduction, and prolificacy [3, 4]. To contain these negative results from happening, it is necessary to prevent brucellosis transmission in animals. Vaccination is often the first preferred way for transmissible diseases, and animals can be vaccinated against brucellosis. Still, it is plenty challenging to eradicate the infection by only vaccination [5]. Detection and elimination of infected animals are other significant measures. It is necessary to understand the dynamics of brucellosis in order to apply additional preventive measures.

Epidemiological models can analyze the course of the diseases since they are constructed by considering the characteristics of the infection and the nature of each one [6–8]. On the contrary, fractional derivatives have allowed many real-world problems to be solved, as they can fulfill complex manners due to their definition [9–12]. Therefore, epidemiological models are analyzed according to the transmission dynamics behavior when combined with fractional derivatives [13–20]. In this context, various mathematical models are discussed by researchers for predicting the dynamics of brucellosis, both integer and fractional. Li et al. examined the impact of preventative strategies and different incidence rates on the transmission of *Brucella* in China via integer-order models [21, 22].

Lolika et al. and Nyerere et al. proposed integer-order dynamical models for the spreading of brucellosis, incorporating the impacts of seasonality [23, 24]. In addition, Nyerere et al. investigated the efficacy of treatment for humans by presenting another integer-order model for brucellosis spread among humans and animals [25]. Sun et al. offered a systematic examination of the transmission dynamics of several brucellosis integer-order mathematical models with their application in China [26]. Lolika and Helikumi proposed and studied two integer-order mathematical models for human brucellosis transmission, in which humans become infected through contact with wildlife and cattle [27]. Another important research topic that has attracted attention recently is the interspecies transmission of brucellosis. Ma et al. posed a discrete model for sheep-human brucellosis transmission dynamics in Jilin, China, and investigated the effectiveness of control measures [28]. Abagna et al. developed a deterministic model to investigate the transmission dynamics and control of bovine brucellosis in a cattle herd [29]. Thongtha and Modnak formulated an interactive bison–human environment mathematical model that contains the impact of human transmission, chronic brucellosis, and control strategy on the brucellosis dynamics [30]. More than that, Peter constructed the fractional model based on hypothetical data, only considering the transmission of brucellosis among cows [31]. Loika and Helikumi describe a fractional model that reveals the transmission of brucellosis among sheep only, utilizing real data from Egypt [32].

As mentioned above, few studies in the literature discuss the fractional-order brucellosis model. Unfortunately, these models do not deal with the transmission of brucellosis among different species of populations. However, since brucellosis increases by showing an exponential behavior depending on the transmission rate of the infection [33], it would be more realistic to study it with a derivative operator that had this behavior. Thanks to the crossover property of the Atangana-Baleanu derivative in terms of Caputo (ABC), it not only allows the explanation of more complex nonlinear phenomena but also does not cause singularity problems at the beginning and end of biological processes. This provides a better insight into the models at their critical points. Due to all these advantages, the ABC derivative has been quite successful in modeling infectious diseases under reality in recent years [34–36].

Incidence rates are another significant element in epidemiological models because they characterize the functional relationship between susceptible and infected people. The incidence rate in epidemiological models is directly proportional to the population's lifestyle and overpopulation and is a consideration that especially impacts the dynamics of transmission. To illustrate the standard incidence rate is expressed based on the total population, while the bi-linear incidence

rate is associated with the law of mass action [37]. The saturated incidence rate improved by Capasso and Serio [38] is frequently favored to describe indirect transmission from bacterial infections such as cholera and brucellosis [22, 26, 39–41], as it causes a saturation level when the exposed (infected) individuals achieve their utmost. Since the amount of Brucella that causes infection by interacting is present at this incidence rate, the interaction rate is controlled with fitting parameters by determining the behavioral change and population density of infected individuals. For bacterial diseases, the saturated incidence rate appears to be closer to reality than the bilinear incidence rate.

Motivated by continuing investigation into this topic, in this study, the deficiencies in the transmission model of brucellosis between sheep and humans proposed by Hou et al. with data from the Inner Mongolia region of China are dealt with and discussed via ABC derivative [42]. As far as we know, the interspecific brucellosis transmission model with ABC derivative has not yet been examined. It is also considered the recovered compartment (with recovery rate) for the human population in the model, the recruitment rate for susceptible humans, and the mortality rate for all humans. Besides, indirect brucellosis transmission is given with a saturated incidence rate instead of the bilinear incidence rate for a more realistic analysis. The unit consistency of the model is also included. It is not always possible to achieve the exact solution of nonlinear fractional-order models. On that account, according to the fixed point theory, it is shown that there exists a unique solution to the fractional brucellosis model. Then, the Adam-type predictor-corrector method is utilized to perform the numerical solution of the model.

The remainder of this article is structured as follows: **Section 2** presents some primary concepts of fractional calculus and the developed interspecies fractional-order brucellosis transmission model. **Section 3** employs fixed-point theory to demonstrate the existence and uniqueness of the model of solutions. **Section 4** is devoted to the numerical solution and discussion. Finally, **Section 5** includes conclusions of the analysis work and gives future direction.

Preliminaries

Here, more details and some definitions and concepts necessary for the completion of this study are presented.

Definition 1 ([43]) For $0 \leq \alpha \leq 1$ and $f \in H^1(a, b)$, the α -order left and right ABC fractional derivatives of the f are expressed as

$${}^{ABC}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{df(\varrho)}{d\varrho} E_\alpha \left[-\frac{\alpha}{1-\alpha} (t-\varrho)^\alpha \right] d\varrho, \quad (1)$$

$${}^{ABC}D_b^\alpha f(t) = \frac{-M(\alpha)}{1-\alpha} \int_t^b \frac{df(\varrho)}{d\varrho} E_\alpha \left[-\frac{\alpha}{1-\alpha} (\varrho-t)^\alpha \right] d\varrho, \quad (2)$$

where E_α is the Mittag-Leffler function and $M(\alpha)$ is the normalization function such as $M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ with $M(0) = M(1) = 1$.

Definition 2 ([43]) The Atangana-Baleanu fractional integral of α -order of the f is expressed as

$${}^{ABC}I_t^\alpha f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_a^t (t-\varrho)^{\alpha-1} f(\varrho) d\varrho. \quad (3)$$

2 Fractional-order brucellosis transmission model

In this section, not only the time derivative is replaced within the ABC sense to examine the brucellosis model by the transmission behavior, but it is also developed by eliminating the deficiencies in the model introduced by Huo et al. [42] for interspecies brucellosis transmission. The brucellosis model as introduced in [42] is given by

$$\left\{ \begin{array}{l} \frac{dS}{dt} = A - \beta S(E + I) - \phi SB - (\mu + \nu)S + \delta V, \\ \frac{dV}{dt} = \nu S - (\mu + \delta)V - \varepsilon\beta V(E + I) - \varepsilon\phi VB, \\ \frac{dE}{dt} = \beta(S + \varepsilon V)(E + I) + \phi(S + \varepsilon V)B - (\sigma + \mu)E, \\ \frac{dI}{dt} = \sigma E - (\mu + c)I, \\ \frac{dB}{dt} = k(E + I) - (d + n\tau)B, \\ \frac{dS_h}{dt} = -\beta_h S_h(E + I) - \phi_h S_h B + \sigma_h(1 - p)I_{ah}, \\ \frac{dI_{ah}}{dt} = \beta_h S_h(E + I) + \phi_h S_h B - \sigma_h I_{ah}, \\ \frac{dI_{ch}}{dt} = \sigma_h p I_{ah}. \end{array} \right. \quad (4)$$

In model (4), susceptible sheep are denoted S , vaccinated sheep by V , exposed sheep by E , and infected sheep by I . The total number of sheep is $N = S + V + E + I$. The number of infectious units in the environment is represented by B . Susceptible humans are signified by S_h , acute infection humans by I_{ah} , and chronic infection humans by I_{ch} . Acute infection humans, if they do not recover, pass into the compartment chronic infection. Total human population $N_h = S_h + I_{ah} + I_{ch}$. The birth and mortality rates for sheep are A and μ , respectively. Susceptible sheep interact with exposed and infected sheep at the rate of β . Brucella is transmitted to susceptible sheep at a rate ϕ . Also, Susceptible sheep are vaccinated at the rate ν , whereas ε is an incorrect vaccination rate. The immunity of vaccinated sheep is lost at a rate of δ . Exposed sheep are infected at the rate σ , and infected sheep are culled at a rate c . Brucella from exposed and infected sheep are shed at a rate k . While Brucella decays in the environment at a rate d , it is effectively disinfected at a rate τ . No data are reported on the transmission among humans of brucellosis between 2005 and 2010, so the rate of transmission among humans is assumed to be zero. Additionally, in the model (4), brucellosis transmission humans directly from sheep at a rate of β_h and indirectly from Brucella at a rate of ϕ_h . Acute infection humans become chronic infection at a rate $\sigma_h p$. Chronic infection humans are also susceptible to the $\sigma_h(1 - p)$ rate. Since the study of Kermack and McKendrick [44], epidemiological models have been created with the recovered compartment to observe the spread of the infection in more detail. However, in model (4), there is no recovered compartment for humans. Additionally, human birth and death rates are other important parameters to consider when analyzing the population. Note that these parameters are not adapted to model (4). For this reason, the model is developed by considering the mentioned deficiencies and is introduced with the ABC derivative so that it can be discussed realistically.

The classical derivative has the s^{-1} second dimension to represent s seconds, while the ${}^{ABC}_0 D_t^\alpha$ fractional derivative has the $s^{-\alpha}$ dimension. The auxiliary parameter θ with second dimension s is employed for unit consistency [45]. The following is the fractional-order brucellosis transmission

model for $0 \leq \alpha \leq 1$ and $t \geq 0$:

$$\begin{cases} \theta^{\alpha-1} ABC_0 D_t^\alpha S = \Lambda_s - \beta S (E + I) - \phi \frac{SB}{\rho+B} - (\mu + \nu) S + \delta V, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha V = \nu S - (\mu + \delta) V - \varepsilon \beta V (E + I) - \varepsilon \phi \frac{VB}{\rho+B}, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha E = \beta (S + \varepsilon V) (E + I) + \phi \frac{(S+\varepsilon V)B}{\rho+B} - (\sigma + \mu) E, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha I = \sigma E - (\mu + c) I, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha B = k (E + I) - (d + n\tau) B, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha S_h = \Lambda_h - \beta_h S_h (E + I) - \phi_h \frac{S_h B}{\rho+B} - \mu_h S_h, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha I_{ah} = \beta_h S_h (E + I) + \phi_h \frac{S_h B}{\rho+B} - (\sigma_h + \mu_h) I_{ah}, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha I_{ch} = \sigma_h p I_{ah} - (\gamma_{ch} + \mu_h) I_{ch}, \\ \theta^{\alpha-1} ABC_0 D_t^\alpha R_h = \sigma_h (1 - p) I_{ah} + \gamma_{ch} I_{ch} - \mu_h R_h, \end{cases} \tag{5}$$

where the initial conditions are

$$\begin{cases} S(0) = S_0, V(0) = V_0, I(0) = I_0, E(0) = E_0, B(0) = B_0, \\ S_h(0) = S_{h0}, I_{ah}(0) = I_{ah0}, I_{ch}(0) = I_{ch0}, R_h(0) = R_{h0}. \end{cases} \tag{6}$$

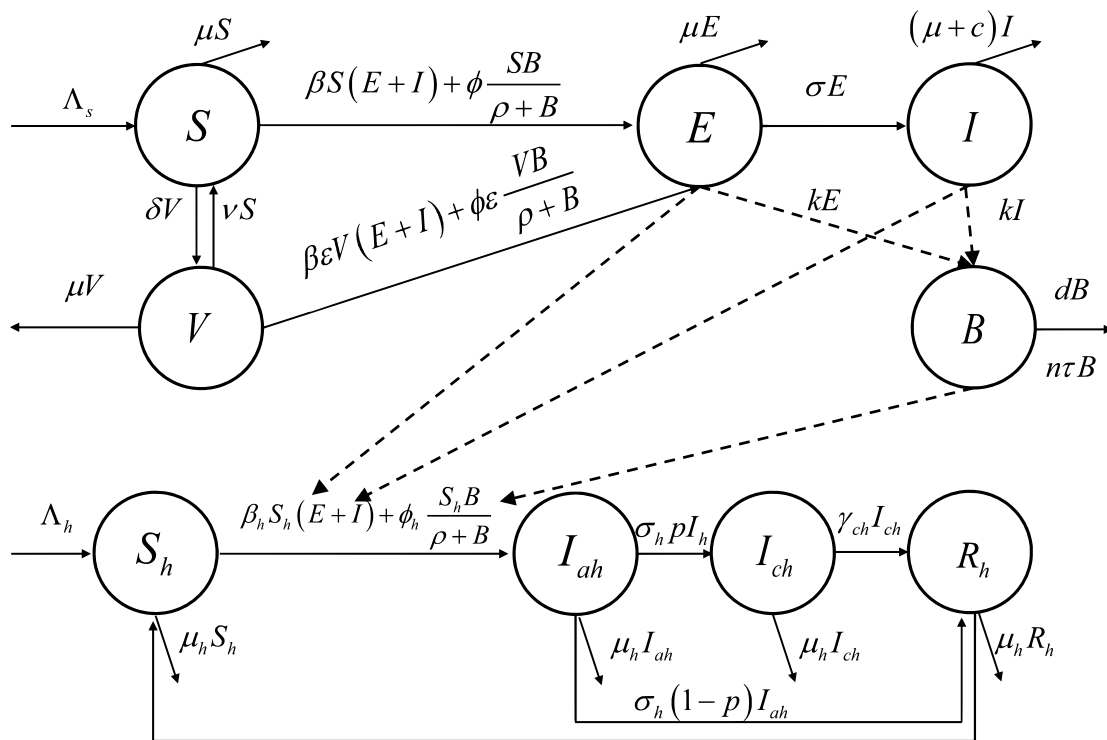


Figure 1. Flowchart of the direct and indirect transmission of the developed fractional brucellosis model

As seen in [Figure 1](#), the human population is separated into four classifications based on their epidemiological stages. The compartment R_h in the model (5) describes recovered humans. To elaborately investigate these stages, it is assumed that there is a recruitment rate Λ_h for compartment S and a natural death rate μ_h for all compartments. Acute and chronic infections are recovered with rates $\sigma_h (1 - p)$ and γ_{ch} , respectively. Also, susceptible sheep and humans catch infections indirectly at rates $\phi \frac{B}{\rho+B}$ and $\phi_h \frac{B}{\rho+B}$, in which ϕ and ϕ_h represent interaction rates with the bacteria-contaminated environment, ρ is the half-saturation constant of the Brucella

population and $\frac{1}{\rho+B}$ is the inhibition effect as well. Hence, $\frac{B}{\rho+B}$ is the probability of exposed sheep and acute infection in humans owning the infection with symptoms, given interaction with the contaminated environment. Although the saturated incidence rate initially behaves linearly, the probability of infection continues to increase to a specific level as the bacteria reproduce. That is to say that even if Brucella in a contaminated environment is in extraordinary numbers, individuals indirectly interacting with more Brucella will not significantly augment the risk to their already threatened health. In this way, the infection transmission level will saturate at rates ϕ and ϕ_h . The amount of Brucella interacted to begin infection is already within this incidence rate. Thus, the saturated incidence rate appears more sensible than the bilinear incidence rate for indirect transmission of bacterial diseases as it prevents the interaction rate from achieving extraordinary numbers by determining appropriate parameters based on the behavioral change and population density of exposed (infected) individuals.

3 Existence and uniqueness

In this section, the existence of a unique solution for the developed brucellosis model is analyzed. By fixed point theory, it is verified that developed fractional transmission dynamics have a unique solution with the initial conditions (6). Suppose a continuous $\mathbb{R} \rightarrow \mathbb{R}$ function described by $\mathcal{H}(\mathcal{J})$ including the sup norm characteristic is a Banach space on $\mathcal{J} = [0, T]$ and

$$\mathcal{Q} = \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}) \times \mathcal{H}(\mathcal{J}),$$

with norm $\|(S, V, E, I, B, S_h, I_{ah}, I_{ch}, R_h)\| = \|S\| + \|V\| + \|E\| + \|I\| + \|B\| + \|S_h\| + \|I_{ah}\| + \|I_{ch}\| + \|R_h\|$, where $\|S\| = \sup_{t \in \mathcal{J}} |S|$, $\|V\| = \sup_{t \in \mathcal{J}} |V(t)|$, $\|E\| = \sup_{t \in \mathcal{J}} |E(t)|$, $\|I\| = \sup_{t \in \mathcal{J}} |I(t)|$, $\|B\| = \sup_{t \in \mathcal{J}} |B(t)|$, $\|S_h\| = \sup_{t \in \mathcal{J}} |S_h(t)|$, $\|I_{ah}\| = \sup_{t \in \mathcal{J}} |I_{ah}(t)|$, $\|I_{ch}\| = \sup_{t \in \mathcal{J}} |I_{ch}(t)|$, $\|R_h\| = \sup_{t \in \mathcal{J}} |R_h(t)|$. Implementing the fractional integral described in (3) to each side of Eq. (5), the model is written below:

$$\begin{aligned} S(t) - S(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \left\{ \Lambda_s - \beta S(t) (E(t) + I(t)) - \phi \frac{S(t)B(t)}{\rho+B(t)} - (\mu + \nu) S(t) + \delta V(t) \right\}, \\ V(t) - V(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \left\{ \nu S(t) - (\mu + \delta) V(t) - \varepsilon \beta V(t) (E(t) + I(t)) - \varepsilon \phi \frac{V(t)B(t)}{\rho+B(t)} \right\}, \\ E(t) - E(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \left\{ \beta (S(t) + \varepsilon V(t)) (E(t) + I(t)) + \phi \frac{(S(t) + \varepsilon V(t))B(t)}{\rho+B(t)} - (\sigma + \mu) E(t) \right\}, \\ I(t) - I(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \{ \sigma E(t) - (\mu + c) I(t) \}, \\ B(t) - B(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \{ k (E(t) + I(t)) - (d + n\tau) B(t) \}, \\ S_h(t) - S_h(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \left\{ \Lambda_h - \beta_h S_h(t) (E(t) + I(t)) - \phi_h \frac{S_h(t)B(t)}{\rho+B(t)} - \mu_h S_h(t) \right\}, \\ I_{ah}(t) - I_{ah}(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \left\{ \beta_h S_h(t) (E(t) + I(t)) + \phi_h \frac{S_h(t)B(t)}{\rho+B(t)} - (\sigma_h + \mu_h) I_{ah}(t) \right\}, \\ I_{ch}(t) - I_{ch}(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \{ \sigma_h p I_{ah}(t) - (\gamma_{ch} + \mu_h) I_{ch}(t) \}, \\ R_h(t) - R_h(0) &= \theta^{1-\alpha} {}^{ABC}_0 I_t^\alpha \{ \sigma_h (1 - p) I_{ah}(t) + \gamma_{ch} I_{ch}(t) - \mu_h R_h(t) \}. \end{aligned} \tag{7}$$

Exerting the definition given by (3), the following expression is acquired:

$$\begin{aligned} S(t) - S(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_1(t, S) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t - \varrho)^{\alpha-1} F_1(\varrho, S) d\varrho, \\ V(t) - V(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_2(t, V) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t - \varrho)^{\alpha-1} F_2(\varrho, V) d\varrho, \\ E(t) - E(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_3(t, E) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t - \varrho)^{\alpha-1} F_3(\varrho, E) d\varrho, \end{aligned}$$

$$\begin{aligned}
 I(t) - I(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_4(t, I) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_4(\varrho, I) d\varrho, \\
 B(t) - B(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_5(t, B) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_5(\varrho, B) d\varrho, \\
 S_h(t) - S_h(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_6(t, S_h) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_6(\varrho, S_h) d\varrho, \\
 I_{ah}(t) - I_{ah}(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_7(t, I_{ah}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_7(\varrho, I_{ah}) d\varrho, \\
 I_{ch}(t) - I_{ch}(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_8(t, I_{ch}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_8(\varrho, I_{ch}) d\varrho, \\
 R_h(t) - R_h(0) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_9(t, R_h) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_9(\varrho, R_h) d\varrho,
 \end{aligned} \tag{8}$$

where the kernels are described by

$$\begin{aligned}
 F_1(t, S) &= \Lambda_s - \beta S(t) (E(t) + I(t)) - \phi \frac{S(t)B(t)}{\rho+B(t)} - (\mu + \nu) S(t) + \delta V(t), \\
 F_2(t, V) &= \nu S(t) - (\mu + \delta) V(t) - \varepsilon\beta V(t) (E(t) + I(t)) - \varepsilon\phi \frac{V(t)B(t)}{\rho+B(t)}, \\
 F_3(t, E) &= \beta (S(t) + \varepsilon V(t)) (E(t) + I(t)) + \phi \frac{(S(t)+\varepsilon V(t))B(t)}{\rho+B(t)} - (\sigma + \mu) E(t), \\
 F_4(t, I) &= \sigma E(t) - (\mu + c) I(t), \\
 F_5(t, B) &= k (E(t) + I(t)) - (d + n\tau) B(t), \\
 F_6(t, S_h) &= \Lambda_h - \beta_h S_h(t) (E(t) + I(t)) - \phi_h \frac{S_h(t)B(t)}{\rho+B(t)} - \mu_h S_h(t), \\
 F_7(t, I_{ah}) &= \beta_h S_h(t) (E(t) + I(t)) + \phi_h \frac{S_h(t)B(t)}{\rho+B(t)} - (\sigma_h + \mu_h) I_{ah}(t), \\
 F_8(t, I_{ch}) &= \sigma_h p I_{ah}(t) - (\gamma_{ch} + \mu_h) I_{ch}(t), \\
 F_9(t, R_h) &= \sigma_h (1 - p) I_{ah}(t) + \gamma_{ch} I_{ch}(t) - \mu_h R_h(t).
 \end{aligned} \tag{9}$$

Theorem 1 *If the below inequality holds*

$$0 \leq \beta (\eta_3 + \eta_4) + \phi \frac{\eta_5}{\rho + \eta_5} < 1,$$

so the kernel F_1 provides for Lipschitz condition and contraction.

Proof Assumed S and S_1 are two functions, the undermentioned inequality is obtained:

$$\begin{aligned}
 \|F_1(t, S) - F_1(t, S_1)\| &= \left\| \Lambda_s - \left(\beta (E(t) + I(t)) + \phi \frac{B(t)}{\rho+B(t)} + \mu + \nu \right) S(t) + \delta V(t) \right. \\
 &\quad \left. - \left(\Lambda_s - \left(\beta (E(t) + I(t)) + \phi \frac{B(t)}{\rho+B(t)} + \mu + \nu \right) S_1(t) + \delta V(t) \right) \right\| \\
 &\leq \left(\beta (E(t) + I(t)) + \phi \frac{B(t)}{\rho+B(t)} + \mu + \nu \right) \|S(t) - S_1(t)\|.
 \end{aligned}$$

Let $\omega_1 = \left(\beta (E(t) + I(t)) + \phi \frac{B(t)}{\rho+B(t)} + \mu + \nu \right)$, where $\|S\| \leq \eta_1, \|V\| \leq \eta_2, \|E\| \leq \eta_3, \|I\| \leq \eta_4, \|B\| \leq \eta_5, \|S_h\| \leq \eta_6, \|I_{ah}\| \leq \eta_7, \|I_{ch}\| \leq \eta_8,$ and $\|R_h\| \leq \eta_9,$ are bounded functions, we have

$$\begin{aligned}
 \|F_1(t, S) - F_1(t, S_1)\| &\leq \left(\beta (\eta_3 + \eta_4) + \phi \frac{\eta_5}{\rho + \eta_5} + \mu + \nu \right) \|S(t) - S_1(t)\| \\
 &\leq \omega_1 \|S(t) - S_1(t)\|.
 \end{aligned}$$

Thus, $F_1(t, S)$ supplies the Lipschitz condition with Lipschitz constant

$$\omega_1 = \left(\beta (E (t) + I (t)) + \phi \frac{B(t)}{\rho + B (t) } + \mu + \nu \right).$$

Furthermore, $0 \leq \omega_1 < 1$, then kernel $F_1(t, S)$ is a contraction. In the same manner, the Lipschitz condition and contraction are provided by the kernels $F_2, F_3, F_4, F_5, F_6, F_7, F_8$, and F_9 given below:

$$\begin{aligned} \|F_2(t, V) - F_2(t, V_1)\| &\leq \omega_2 \|V(t) - V_1(t)\|, \\ \|F_3(t, E) - F_3(t, E_1)\| &\leq \omega_3 \|E(t) - E_1(t)\|, \\ \|F_4(t, I) - F_4(t, I_1)\| &\leq \omega_4 \|I(t) - I_1(t)\|, \\ \|F_5(t, B) - F_5(t, B_1)\| &\leq \omega_5 \|B(t) - B_1(t)\|, \\ \|F_6(t, S_h) - F_6(t, S_{h1})\| &\leq \omega_6 \|S_h(t) - S_{h1}(t)\|, \\ \|F_7(t, I_{ah}) - F_7(t, I_{ah1})\| &\leq \omega_7 \|I_{ah}(t) - I_{ah1}(t)\|, \\ \|F_8(t, I_{ch}) - F_8(t, I_{ch1})\| &\leq \omega_8 \|I_{ch}(t) - I_{ch1}(t)\|, \\ \|F_9(t, R_h) - F_9(t, R_{h1})\| &\leq \omega_9 \|R_h(t) - R_{h1}(t)\|. \end{aligned} \tag{10}$$

The kernels in Eq. (8) can be rewritten as:

$$\begin{aligned} S(t) &= S(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_1(t, S) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_1(\varrho, S) d\varrho, \\ V(t) &= V(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_2(t, V) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_2(\varrho, V) d\varrho, \\ E(t) &= E(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_3(t, E) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_3(\varrho, E) d\varrho, \\ I(t) &= I(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_4(t, I) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_4(\varrho, I) d\varrho, \\ B(t) &= B(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_5(t, B) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_5(\varrho, B) d\varrho, \\ S_h(t) &= S_h(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_6(t, S_h) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_6(\varrho, S_h) d\varrho, \\ I_{ah}(t) &= I_{ah}(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_7(t, I_{ah}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_7(\varrho, I_{ah}) d\varrho, \\ I_{ch}(t) &= I_{ch}(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_8(t, I_{ch}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_8(\varrho, I_{ch}) d\varrho, \\ R_h(t) &= R_h(0) + \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_9(t, R_h) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_9(\varrho, R_h) d\varrho. \end{aligned} \tag{11}$$

Going recursively, Eq. (11) yielded

$$\begin{aligned} S_n(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_1(t, S_{n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_1(\varrho, S_{n-1}) d\varrho, \\ V_n(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_2(t, V_{n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_2(\varrho, V_{n-1}) d\varrho, \\ E_n(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_3(t, E_{n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_3(\varrho, E_{n-1}) d\varrho, \end{aligned}$$

$$\begin{aligned}
I_n(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_4(t, I_{n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_4(\varrho, I_{n-1}) d\varrho, \\
B_n(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_5(t, B_{n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_5(\varrho, B_{n-1}) d\varrho, \\
S_{h,n}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_6(t, S_{h,n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_6(\varrho, S_{h,n-1}) d\varrho, \\
I_{ah,n}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_7(t, I_{ah,n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_7(\varrho, I_{ah,n-1}) d\varrho, \\
I_{ch,n}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_8(t, I_{ch,n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_8(\varrho, I_{ch,n-1}) d\varrho, \\
R_{h,n}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} F_9(t, R_{h,n-1}) + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} F_9(\varrho, R_{h,n-1}) d\varrho,
\end{aligned} \tag{12}$$

along with the initial conditions $S(0) = S_0$, $V(0) = V_0$, $I(0) = I_0$, $E(0) = E_0$, $B(0) = B_0$, $S_h(0) = S_{h0}$, $I_{ah}(0) = I_{ah0}$, $I_{ch}(0) = I_{ch0}$, and $R_h(0) = R_{h0}$. By taking the difference between successive terms, the following equalities are reached:

$$\begin{aligned}
\psi_{1n}(t) = S_n(t) - S_{n-1}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_1(t, S_{n-1}) - F_1(t, S_{n-2})\} \\
&+ \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_1(\varrho, S_{n-1}) - F_1(\varrho, S_{n-2})\} d\varrho,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\psi_{2n}(t) = V_n(t) - V_{n-1}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_2(t, V_{n-1}) - F_2(t, V_{n-2})\} \\
&+ \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_2(\varrho, V_{n-1}) - F_2(\varrho, V_{n-2})\} d\varrho,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\psi_{3n}(t) = E_n(t) - E_{n-1}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_3(t, E_{n-1}) - F_3(t, E_{n-2})\} \\
&+ \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_3(\varrho, E_{n-1}) - F_3(\varrho, E_{n-2})\} d\varrho,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\psi_{4n}(t) = I_n(t) - I_{n-1}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_4(t, I_{n-1}) - F_4(t, I_{n-2})\} \\
&+ \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_4(\varrho, I_{n-1}) - F_4(\varrho, I_{n-2})\} d\varrho,
\end{aligned} \tag{16}$$

$$\begin{aligned}
\psi_{5n}(t) = B_n(t) - B_{n-1}(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_5(t, B_{n-1}) - F_5(t, B_{n-2})\} \\
&+ \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_5(\varrho, B_{n-1}) - F_5(\varrho, B_{n-2})\} d\varrho,
\end{aligned} \tag{17}$$

$$\begin{aligned} \psi_{6n}(t) &= S_{h,n}(t) - S_{h,n-1}(t) = \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_6(t, S_{h,n-1}) - F_6(t, S_{h,n-2})\} \\ &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_6(\varrho, S_{h,n-1}) - F_6(\varrho, S_{h,n-2})\} d\varrho, \end{aligned} \tag{18}$$

$$\begin{aligned} \psi_{7n}(t) &= I_{ah,n}(t) - I_{ah,n-1}(t) = \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_7(t, I_{ah,n-1}) - F_7(t, I_{ah,n-2})\} \\ &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_7(\varrho, I_{ah,n-1}) - F_7(\varrho, I_{ah,n-2})\} d\varrho, \end{aligned} \tag{19}$$

$$\begin{aligned} \psi_{8n}(t) &= I_{ch,n}(t) - I_{ch,n-1}(t) = \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_8(t, I_{ch,n-1}) - F_8(t, I_{ch,n-2})\} \\ &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_8(\varrho, I_{ch,n-1}) - F_8(\varrho, I_{ch,n-2})\} d\varrho, \end{aligned} \tag{20}$$

$$\begin{aligned} \psi_{9n}(t) &= R_{h,n}(t) - R_{h,n-1}(t) = \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_9(t, R_{h,n-1}) - F_9(t, R_{h,n-2})\} \\ &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_9(\varrho, R_{h,n-1}) - F_9(\varrho, R_{h,n-2})\} d\varrho. \end{aligned} \tag{21}$$

In addition, it is obvious that $S_n(t) = \sum_{i=0}^n \psi_{1i}(t)$, $V_n(t) = \sum_{i=0}^n \psi_{2i}(t)$, $E_n(t) = \sum_{i=0}^n \psi_{3i}(t)$, $I_n(t) = \sum_{i=0}^n \psi_{4i}(t)$, $B_n(t) = \sum_{i=0}^n \psi_{5i}(t)$, $S_{h,n}(t) = \sum_{i=0}^n \psi_{6i}(t)$, $I_{ah,n}(t) = \sum_{i=0}^n \psi_{7i}(t)$, $I_{ch,n}(t) = \sum_{i=0}^n \psi_{8i}(t)$, and $R_{h,n}(t) = \sum_{i=0}^n \psi_{9i}(t)$. By implementing the norm to both sides of Eq. (13) and utilizing the triangular inequality, it can be expressed as:

$$\begin{aligned} \|\psi_{1n}(t)\| &= \|S_n(t) - S_{n-1}(t)\| = \left\| \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_1(t, S_{n-1}) - F_1(t, S_{n-2})\} \right. \\ &\quad \left. + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_1(\varrho, S_{n-1}) - F_1(\varrho, S_{n-2})\} d\varrho \right\| \\ &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \|F_1(t, S_{n-1}) - F_1(t, S_{n-2})\| \\ &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \|F_1(\varrho, S_{n-1}) - F_1(\varrho, S_{n-2})\| d\varrho. \end{aligned} \tag{22}$$

Due to the fact that the Lipschitz condition is provided for by kernel F_1 , the following can be written:

$$\begin{aligned} \|\psi_{1n}(t)\| &= \|S_n(t) - S_{n-1}(t)\| \leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \omega_1 \|S_{n-1} - S_{n-2}\| \\ &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \omega_1 \int_0^t (t-\varrho)^{\alpha-1} \|S_{n-1} - S_{n-2}\| d\varrho, \end{aligned} \tag{23}$$

then obtained as:

$$\|\psi_{1n}(t)\| \leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \omega_1 \|\psi_{1(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \omega_1 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{1(n-1)}(\varrho)\| d\varrho. \tag{24}$$

Similarly, the following results are acquired

$$\begin{aligned}
 \|\psi_{2n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_2 \|\psi_{2(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_2 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{2(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{3n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_3 \|\psi_{3(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_3 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{3(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{4n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_4 \|\psi_{4(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_4 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{4(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{5n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_5 \|\psi_{5(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_5 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{5(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{6n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_6 \|\psi_{6(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_6 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{6(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{7n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_7 \|\psi_{7(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_7 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{7(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{8n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_8 \|\psi_{8(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_8 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{8(n-1)}(\varrho)\| d\varrho, \\
 \|\psi_{9n}(t)\| &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_9 \|\psi_{9(n-1)}(t)\| + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_9 \int_0^t (t-\varrho)^{\alpha-1} \|\psi_{9(n-1)}(\varrho)\| d\varrho.
 \end{aligned} \tag{25}$$

Considering the results received, the existence of the solution of model (5) is given with the help of the following theorem.

Theorem 2 *If there exists t_0 satisfying the following inequality*

$$\frac{1-\alpha}{M(\alpha)}\omega_i + \frac{t_0}{M(\alpha)\Gamma(\alpha)}\omega_i < 1, \text{ for } i = 1, 2, \dots, 9, \tag{26}$$

then the model (5) has a solution.

Proof It is established that the functions $S(t), V(t), I(t), E(t), B(t), S_h(t), I_{ah}(t), I_{ch}(t)$, and $R_h(t)$ are bounded, and their kernels are fulfilled the Lipschitz condition. Employing the recursive technique, the following relationship is achieved:

$$\begin{aligned}
 \|\psi_{1n}(t)\| &\leq \|S(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_1 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_1 \right]^n, \\
 \|\psi_{2n}(t)\| &\leq \|V(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_2 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_2 \right]^n, \\
 \|\psi_{3n}(t)\| &\leq \|E(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_3 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_3 \right]^n, \\
 \|\psi_{4n}(t)\| &\leq \|I(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_4 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_4 \right]^n, \\
 \|\psi_{5n}(t)\| &\leq \|B(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_5 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_5 \right]^n, \\
 \|\psi_{6n}(t)\| &\leq \|S_h(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_6 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_6 \right]^n, \\
 \|\psi_{7n}(t)\| &\leq \|I_{ah}(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_7 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_7 \right]^n, \\
 \|\psi_{8n}(t)\| &\leq \|I_{ch}(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_8 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_8 \right]^n, \\
 \|\psi_{9n}(t)\| &\leq \|R_h(0)\| \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_9 + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_9 \right]^n.
 \end{aligned} \tag{27}$$

Hence, the solutions exist and are provided continuously for the model (5). For the sake of clarity,

to exhibit that the functions $S(t), V(t), I(t), E(t), B(t), S_h(t), I_{ah}(t), I_{ch}(t)$, and $R_h(t)$ have a solution to model (5), suppose that

$$\begin{aligned}
 S(t) - S(0) &= S_n(t) - \Phi_{1n}(t), \\
 V(t) - V(0) &= V_n(t) - \Phi_{2n}(t), \\
 E(t) - E(0) &= E_n(t) - \Phi_{3n}(t), \\
 I(t) - I(0) &= I_n(t) - \Phi_{4n}(t), \\
 B(t) - B(0) &= B_n(t) - \Phi_{5n}(t) \\
 S_h(t) - S_h(0) &= S_{h,n}(t) - \Phi_{6n}(t), \\
 I_{ah}(t) - I_{ah}(0) &= I_{ah,n}(t) - \Phi_{7n}(t), \\
 I_{ch}(t) - I_{ch}(0) &= I_{ch,n}(t) - \Phi_{8n}(t), \\
 R_h(t) - R_h(0) &= R_{h,n}(t) - \Phi_{9n}(t).
 \end{aligned}
 \tag{28}$$

Accordingly, the expression $\|\Phi_{1n}(t)\|$ is acquired as:

$$\begin{aligned}
 \|\Phi_{1n}(t)\| &= \left\| \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_1(t, S) - F_1(t, S_{n-1})\} \right. \\
 &\quad \left. + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \{F_1(\varrho, S) - F_1(\varrho, S_{n-1})\} d\varrho \right\| \\
 &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \|F_1(t, S) - F_1(t, S_{n-1})\| \\
 &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-\varrho)^{\alpha-1} \|F_1(\varrho, S) - F_1(\varrho, S_{n-1})\| d\varrho \\
 &\leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \omega_1 \|S - S_{n-1}\| + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \omega_1 \|S - S_{n-1}\|.
 \end{aligned}
 \tag{29}$$

After using this process recursively, then yields at t_0

$$\|\Phi_{1n}(t)\| \leq \left[\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} + \frac{t_0\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \right]^{n+1} \omega_1^{n+1} P_1.
 \tag{30}$$

As n approaches infinity, taking the limit to both sides of Eq. (30), it is obtained as $\|\Phi_{1n}(t)\| \rightarrow 0$. Consequently, the existence of the solution of the model (5) is verified. Similarly, it is found $\|\Phi_{2n}(t)\| \rightarrow 0, \|\Phi_{3n}(t)\| \rightarrow 0, \|\Phi_{4n}(t)\| \rightarrow 0, \|\Phi_{5n}(t)\| \rightarrow 0, \|\Phi_{6n}(t)\| \rightarrow 0, \|\Phi_{7n}(t)\| \rightarrow 0, \|\Phi_{8n}(t)\| \rightarrow 0$, and $\|\Phi_{9n}(t)\| \rightarrow 0$. Now, the uniqueness of the solution is given by the following theorem.

Theorem 3 *The model (5) has a unique solution, provided that*

$$\frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \omega_i + \frac{t_0\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \omega_i < 1, \text{ for } i = 1, 2, \dots, 9.$$

Proof Assumed that $S_1(t), V_1(t), E_1(t), I_1(t), B_1(t), S_{h1}(t), I_{ah1}(t), I_{ch1}(t)$, and $R_{h1}(t)$ are also solutions of the model (5). Then,

$$\begin{aligned}
 S(t) - S_1(t) &= \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)} \{F_1(t, S) - F_1(t, S_1)\} \\
 &\quad + \frac{\alpha\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)} \int_a^t (t-\varrho)^{\alpha-1} \{F_1(\varrho, S) - F_1(\varrho, S_1)\} d\varrho.
 \end{aligned}
 \tag{31}$$

Considering that the kernel fulfills the Lipschitz condition, implementing the norm to both sides

of Eq. (31), it reaches the inequality presented below:

$$\|S(t) - S_1(t)\| \leq \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_1 \|S(t) - S_1(t)\| + \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_1 \|S(t) - S_1(t)\|. \quad (32)$$

It gives

$$\|S(t) - S_1(t)\| \left(1 - \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_1 - \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_1\right) \leq 0. \quad (33)$$

As $\left(1 - \frac{(1-\alpha)\theta^{1-\alpha}}{M(\alpha)}\omega_1 - \frac{t\theta^{1-\alpha}}{M(\alpha)\Gamma(\alpha)}\omega_1\right) > 0$, then $\|S(t) - S_1(t)\| = 0$. As a result, it is attained $S(t) = S_1(t)$. Similarly, it is seen that $V(t) = V_1(t)$, $E(t) = E_1(t)$, $I(t) = I_1(t)$, $B(t) = B_1(t)$, $S_h(t) = S_{h1}(t)$, $I_{ah}(t) = I_{ah1}(t)$, $I_{ch}(t) = I_{ch1}(t)$, and $R_h(t) = R_{h1}(t)$. Thus, it is concluded that model (5) has a unique solution.

4 Numerical solutions and discussion

In this section, numerical solutions of the improved fractional brucellosis model are obtained using the parameter values in Table 1 for $t = 12$ years. The initial conditions are $S(0) = 4.341 \times 10^7$, $V(0) = 8.44 \times 10^6$, $E(0) = 0$, $I(0) = 1.33 \times 10^6$, $B(0) = 6 \times 10^6$, $S_h(0) = 2.384 \times 10^7$, $I_{ah}(0) = 8663$, $I_{ch}(0) = 0$, and $R_h(0) = 0$ [42]. The Adams-type predictor-corrector method is applied to solve the fractional-order brucellosis model [46]. All numerical results are given by means of MATLAB.

Table 1. Interpretation of parameters in model (5)

Parameter	Value & Units & References
Λ_s	11629200 (sheep year ⁻¹), [42]
β	1.48×10^{-8} (sheep ⁻¹ year ⁻¹), [42]
ϕ	1.7×10^{-10} (bacteria ⁻¹ year ⁻¹), [42]
ρ	10^7 (bacteria), Assumed
μ	0.22 (year ⁻¹), [42]
ν	0.316 (year ⁻¹), [42]
δ	0.4 (year ⁻¹), [42]
ε	0.18 (year ⁻¹), [42]
σ	1 (year ⁻¹), [42]
c	0.15 (year ⁻¹), [42]
k	15 (bacteria sheep ⁻¹ year ⁻¹), [42]
d	3.6 (year ⁻¹), [42]
n	0 (year ⁻¹), [42]
τ	0 (year ⁻¹), [42]
Λ_h	0.0057 (human year ⁻¹), [47]
β_h	1.58×10^{-10} (sheep ⁻¹ year ⁻¹), [42]
ϕ_h	1×10^{-11} (bacteria ⁻¹ year ⁻¹), [42]
μ_h	0.0054 (year ⁻¹), [47]
$\sigma_h p$	0.6 (year ⁻¹), [42]
$\sigma_h(1-p)$	0.4 (year ⁻¹), [42]
γ_{ch}	0.5 (year ⁻¹), [48]

Figure 2 compares the developed fractional brucellosis model with the integer model over a 12-year period. The fractional brucellosis model demonstrates a slight reduction in the transmission of bacteria to sheep and humans with a saturated incidence rate. This reduction is observed due

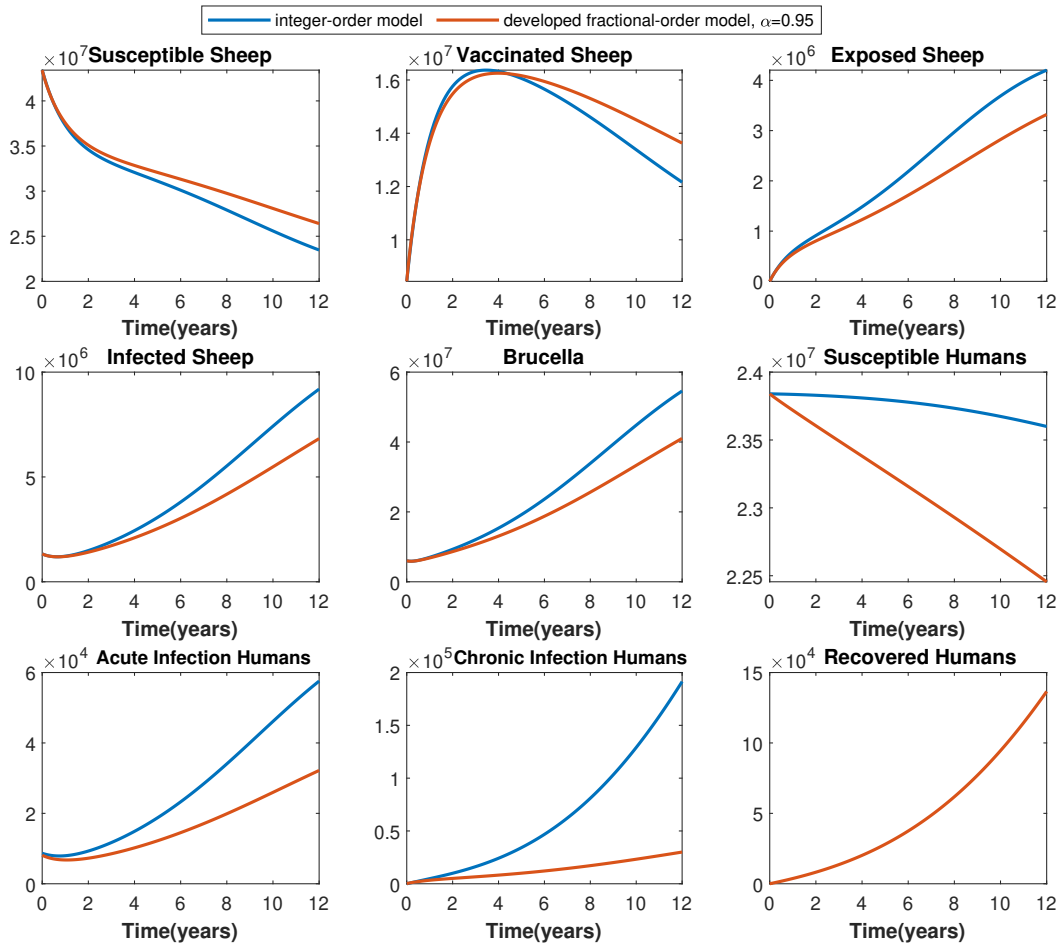


Figure 2. The comparison of the integer-order model with the developed fractional derivative model

to various factors, including the human birth and death rates and the existence of a recovered compartment. The inclusion of a recovered compartment of humans into the model resulted in a reduction in the number of susceptible individuals. In the integer order brucellosis model, chronically infected humans become susceptible upon recovery. In contrast, in the developed fractional brucellosis model, the behavior of susceptible and recovered humans in the process can be clearly examined with the addition of the recovered compartment of humans. This allows for the review of the transmission process with greater specificity. Moreover, a reduction in the rate of transmission is observed due to the continued shedding of *Brucella* bacteria from exposed and infected sheep throughout the process. [Figure 3](#), the developed fractional brucellosis model is compared in different orders and this change is reflected in the graphical results as flexibility. In other words, the behavior appears to fade as the order of derivatives decreases. Nevertheless, it is evident from the graphs that an interspecies brucellosis epidemic has initiated. Consequently, it is imperative to implement various control strategies to eliminate the infection.

5 Conclusions

Brucellosis is an interspecies infectious disease that influences people and animal health, as well as financial growth in affected areas. Hence, figuring out the transmission dynamic of brucellosis has become crucial. For this purpose, a fractional-order model for interspecies transmission of brucellosis was developed. First, the current brucellosis transmission model with integer order

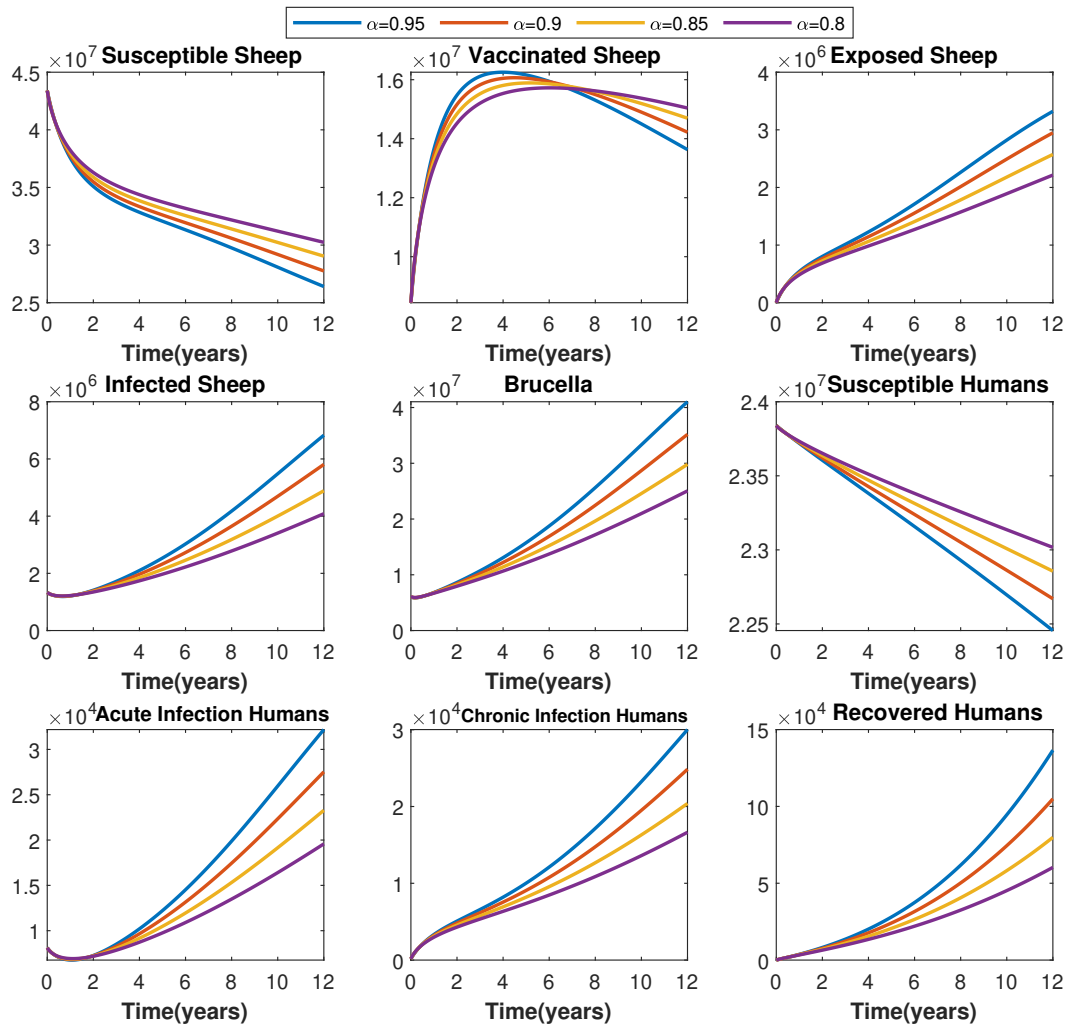


Figure 3. The comparison of the developed fractional derivative model with different fractional orders

was incorporated with the ABC fractional derivative to examine the transmission by its behavior. Then, to observe more specifically, the model was discussed with the recovered compartment, recruitment, and natural mortality rate for humans. Also, the saturated incidence rate was proposed for brucellosis as indirectly transmitted to individuals from the environment. The fixed-point theory was used to reveal the existence and uniqueness of solutions to the developed model. The model has been numerically solved using Adams-type predictor-corrector method with help of the MATLAB. Graphically, the effect of fractional derivatives of different orders on the model behavior was examined. It was observed that the interspecies brucellosis epidemic increased in the process. For this reason, an analysis of the dynamics and sensitivity of the developed fractional brucellosis model is planned for future work. Thus, we hope that the factors that cause brucellosis transmission, which adversely affects the community, could be determined.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

All data generated or analyzed during this study are included in this article.

Ethical approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

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Author's contributions

D.Y.: Investigation, Methodology, Formal Analysis, Writing-Original draft preparation, Visualization, Software. B.B.İ.E.: Project administration, Conceptualization, Validation, Supervision, Writing-Reviewing and Editing, Software. The authors have read and agreed to the published version of the manuscript.

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