



RESEARCH PAPER

Piecewise fractional analysis of the migration effect in plant-pathogen-herbivore interactions

Mati ur Rahman ^{1,2,*,‡}, Muhammad Arfan ^{3,‡} and Dumitru Baleanu ^{4,‡}

¹School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road Shanghai, PR China, ²Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon, ³Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan, ⁴Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Türkiye

* Corresponding Author

‡ matior.rahman@lau.edu.lb (Mati ur Rahman); arfan_uom@yahoo.com (Muhammad Arfan); dumitru@cankaya.edu.tr (Dumitru Baleanu)

Abstract

This study introduces several updated results for the piecewise plant-pathogen-herbivore interactions model with singular-type and nonsingular fractional-order derivatives. A piecewise fractional model has developed to describe the interactions between plants, disease, (insect) herbivores, and their natural enemies. We derive essential findings for the aforementioned problem, specifically regarding the existence and uniqueness of the solution, as well as various forms of Ulam Hyers (U-H) type stability. The necessary results were obtained by utilizing fixed-point theorems established by Schauder and Banach. Additionally, the U-H stabilities were determined based on fundamental principles of nonlinear analysis. To implement the model as an approximate piecewise solution, the Newton Polynomial approximate solution technique is employed. The applicability of the model was validated through numerical simulations both in fractional as well as piecewise fractional format. The motivation of our article is that we have converted the integer order problem to a global piecewise and fractional order model in the sense of Caputo and Atangana-Baleanu operators and investigate it for existence, uniqueness of solution, Stability of solution and approximate solution along with numerical simulation for the validity of our obtained scheme.

Keywords: Crossover behavior; piecewise global fractional derivatives; Newton interpolation formula; equicontinuous mapping; Schauder's fixed point theorem

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1 Introduction

The interactions between plants, herbivores, and pathogens in nature are complex and multifaceted. According to a reference [1], approximately 50% of the estimated 6 million species of insects are

herbivorous. On the other hand, plant pathogenic microbes pose significant threats to plants [2], but their total number has not been accurately estimated. Plants have evolved sophisticated defense mechanisms to detect and respond to multiple attacks by herbivores and pathogens [3]. These mechanisms can be broadly classified into two types: physical and chemical defenses. Regarding chemical defense, plants have the ability to release volatile organic compounds (VOCs) that can attract natural enemies of herbivores. This process can ultimately help to decrease herbivore pressure [4, 5]. Predatory mites and parasitic wasps are among the natural enemies of herbivores that are attracted by volatile organic compounds (VOCs) released by plants [6].

A good example of this is when lima bean and apple plants are attacked by spider mites. In response to this damage, the plants release volatile compounds that can attract predatory mites to help control the spider mite population. Similarly, several plant species, including cucumber, corn, and cotton, release herbivore-induced plant volatiles (HIPVs) when attacked by herbivores. The use of HIPVs can help control pests and potentially reduce the need for artificial pesticides. Plants have the ability to release HIPVs, which are lipophilic liquids with high vapor pressures that can be emitted from various parts of a plant, such as its leaves, flowers, and fruits [7]. Through the release of HIPVs, plants may help protect forestry and agriculture by potentially attracting predatory arthropods or repelling herbivores, ultimately promoting plant fitness [8–10]. In addition, the release of volatile compounds can also help damaged plants by attracting natural enemies of herbivores [11]. Studies focusing on the interactions between plants and herbivores, as well as plants and pathogens, have been extensively researched. The influence of VOCs in mediating tri-trophic interactions, which occur when plants are attacked by both herbivores and pathogens, has yet to be fully understood. In a recent study by Liu et al. [12], they examined a model that involved plants, herbivores, and natural enemies of herbivores in the context of tri-trophic interactions, but did not take into account the potential pathogenic effects on the plant population. The study indicated that an increase in the strength of plant-induced volatile attraction to natural enemies resulted in a greater fluctuation amplitude of plant biomass and herbivore population.

Previous studies have primarily investigated the interactions between plants and either herbivores or pathogens. However, the extent to which VOCs mediate tri-trophic interactions when plants are attacked by both herbivores and pathogens remains uncertain. In a recent study by Liu et al. [12], they investigated tri-trophic interactions involving plants, herbivores, and natural enemies of herbivores without considering the pathogenic effects on plant populations. They found that an increase in the attraction strength of plant-induced volatiles to natural enemies leads to high fluctuation amplitudes of plant biomass and herbivore populations. Based on the study by Fergola and Wang [13], when the attack strength of natural enemies reaches a certain level, the fluctuation amplitude of plant biomass and herbivore population decreases, and the plant biomass tends to approach its environmental carrying capacity. They enhanced Liu et al.'s model [12] by considering the impact of time delay. Their research indicated that for Volterra-type interactions, chemical attractions do not affect the threshold value for the persistence of herbivore and carnivore populations. In addition, their observations showed that the presence of carnivores could lead to a decrease in herbivore density while increasing plant density. Moreover, the model demonstrated a fold bifurcation when the predation process follows Leslie-type interactions.

Based on the literature discussed above, we investigated a model for plants [14] using the framework of piecewise derivative and integral operators. The model consists of four compartments, including the susceptible plant compartment (S), the infected plant compartment (I), and the herbivore (Y) and natural enemy (Z) populations. The system of ordinary differential equations

can be expressed as follows:

$$\begin{aligned}
 \frac{dS}{dt} &= S \left[r \left(1 - \frac{S}{k} \right) - \frac{\beta I}{1 + \alpha S} - p_1 Y \right], \\
 \frac{dI}{dt} &= I \left(\frac{\beta S}{1 + \alpha S} - \omega \right), \\
 \frac{dY}{dt} &= Y (-d_1 + C_1 p_1 S - p_2 Z), \\
 \frac{dZ}{dt} &= -Y_2 Z + C_2 p_2 Y Z + \mu.
 \end{aligned}
 \tag{1}$$

The parameters used in the above system with details are presented in Table 1.

Table 1. Details of the used parameters in model (1).

Notations	Details
r	The growth rate of plant
k	The rate of environmental carrying capacity
β	The rate of transmission of the disease
α	Rate of the inhibition effect
ω	Death rate of the infected plants
d_1	Herbivore death rate
c_1, c_2	The rates of conversion
μ	The rate herbivore natural enemy
p_1, p_2	The rates of predation for plants

Fractional derivatives are known for their ability to incorporate memory and genetic effects, making them essential for studying various real-world dynamic problems. The application of fractional-order derivatives in mathematical models of infectious disease spread makes the situation more realistic and plausible. The conventional fractional calculus method involves long memory effects, which can lead to complications when dealing with long-term calculation problems. Moreover, power-law long memory is represented using classical fractional calculus mathematical tools, which consist of fractional order derivatives and integrals. Several researchers have presented different fractional operators and investigated mathematical modeling of linear and nonlinear problems to discuss real-world phenomena [15–18]. It is worth noting that when modeling different memory phenomena, the memory process is generally split into two stages. The memory process is typically divided into two stages, one involving short-term memory with permanent retention, while the other is governed by a simple fractional derivative model [19]. To tackle the issue of short-term memory, it is necessary to employ equations with piecewise fractional-order derivatives. This approach can enhance performance and efficiency, enabling clearer explanations of physical phenomena (see [20] for further information). Moreover, the idea of using piecewise derivatives of fractional orders has recently been explored in several studies. The concept of piecewise fractional differential equations has been employed by researchers in recent studies, after Atangana and Araz presented in [21], to explore various dynamical systems [22–24]. Such operators have proven useful in solving problems that exhibit crossover behavior and are gaining popularity among researchers. For instance, in [25], the CAT-T cells-SARS-2 disease model was investigated using the concept of piecewise fractional differential equations. Similarly, a Covid-19 model was studied using this approach in [26]. Additionally, authors in [27] examined the third wave of the Covid-19 outbreak in Türkiye, Spain, and Czechia by

employing piecewise differential and integral operators in their model. In addition, the concept of piecewise operators is being utilized by several authors to obtain numerical results for the food web model (see Ref. [28]). Numerous works have been documented by several researchers regarding piecewise differential and integral operators with various operators [29–33].

As for as the contribution of our article is concerned, we have converted the integer order problem to a global piecewise and fractional order model in the sense of Caputo and Atangana Baleanu operators (PCABC). As fractional operators are generalized because they have an extra degree of freedom and choices. Therefore, we checked the dynamics of different fractional orders lying between 0 and 1, and compared them with the integer order. On small fractional orders, stability is achieved quickly. The piecewise fractional model is investigated for the existence and uniqueness of the solution in the sense of fractional Caputo and Atangana Baleanu operator having a kernel of non-singularity in the form of the exponential function. The approximate or semi-analytical solution is obtained by the technique of piecewise fractional Caputo and ABC derivative. Taking different fractional orders we have simulated the obtained scheme for the first four terms. We have also compared the fractional dynamics with the integer order dynamics. All the quantities of the proposed problem are converging to their equilibrium points showing spectrum dynamics with the removal of singularity as well as the crossover or abrupt dynamics.

Motivated by the above-mentioned work, we study the system taken from [14] in the sense of piecewise Caputo and Atangana-Baleanu-Caputo operator as given below:

$$\begin{cases} {}_0^{PCABC}D_t^\nu S(t) = S[r(1 - \frac{S}{k}) - \frac{\beta I}{1+\alpha S} - p_1 Y], \\ {}_0^{PCABC}D_t^\nu I(t) = I(\frac{\beta S}{1+\alpha S} - \omega), \\ {}_0^{PCABC}D_t^\nu Y(t) = Y(-Y_1 + C_1 p_1 S - p_2 Z), \\ {}_0^{PCABC}D_t^\nu Z(t) = -Y_2 Z + C_2 p_2 YZ + \mu. \end{cases} \quad (2)$$

In order to elaborate further, equation (2) can be expressed as follows:

$$\begin{aligned} {}_0^{CABC}D_t^\nu(S(t)) &= \begin{cases} {}_0^C D_t^\nu(S(t)) = {}^C f_1(S, I, Y, Z, t), & 0 < t \leq t_1, \\ {}_0^{ABC} D_t^\nu(S(t)) = {}^{ABC} f_1(S, I, Y, Z, t), & t_1 < t \leq T, \end{cases} \\ {}_0^{CABC}D_t^\nu(I(t)) &= \begin{cases} {}_0^C D_t^\nu(I(t)) = {}^C f_2(S, I, Y, Z, t), & 0 < t \leq t_1, \\ {}_0^{ABC} D_t^\nu(I(t)) = {}^{ABC} f_2(S, I, Y, Z, t), & t_1 < t \leq T, \end{cases} \\ {}_0^{CABC}D_t^\nu(Y(t)) &= \begin{cases} {}_0^C D_t^\nu(Y(t)) = {}^C f_3(S, I, Y, Z, t), & 0 < t \leq t_1, \\ {}_0^{ABC} D_t^\nu(Y(t)) = {}^{ABC} f_3(S, I, Y, Z, t), & t_1 < t \leq T, \end{cases} \\ {}_0^{CABC}D_t^\nu(Z(t)) &= \begin{cases} {}_0^C D_t^\nu(Z(t)) = {}^C f_4(S, I, Y, Z, t), & 0 < t \leq t_1, \\ {}_0^{ABC} D_t^\nu(Z(t)) = {}^{ABC} f_4(S, I, Y, Z, t), & t_1 < t \leq T, \end{cases} \end{aligned} \quad (3)$$

In our study, we used piecewise fractional operators to describe the dynamics of the system, as they can capture crossover and abrupt dynamics effectively [34, 35]. Specifically, we applied different fractional operators to different intervals of the system and provided a qualitative analysis for each subinterval. To assess stability, we used the UH stability analysis. Furthermore, we investigated the utilization of piecewise terms and fractional orders in the final step of the system to obtain approximate solutions that can accommodate dynamics of both integer and rational orders.

The rest of the paper is organized as in the section 2 we include some preliminaries in form of definitions and lemmas. Section 3 comprised with the existence and uniqueness of solution of said model along with the stability of solution. In section 4 we established the numerical scheme

for the approximate solution of model in fractional piecewise format. To show the validness of the obtain scheme we provide the graphical representation in the section 5 of numerical simulation. At the end we give the concluding remarks.

2 Preliminaries

In this section, we recalled some definition from the literature.

Definition 1 [16] The ABC operator of a function $\mathbf{X}(t)$ with the condition $\mathbf{X}(t) \in \mathcal{H}^1(0, T)$ is defined as follows:

$${}^{\text{ABC}}_0D_t^\nu(\mathbf{X}(t)) = \frac{\text{ABC}(\nu)}{1-\nu} \int_0^t \frac{d}{dy} \mathbf{X}(y) E_\nu \left[\frac{-\nu(t-y)^\nu}{1-\nu} \right] dy. \quad (4)$$

Definition 2 [16] If we consider $\mathbf{X}(t) \in \mathcal{H}^1(0, T)$, the ABC integral can be expressed as follows:

$${}^{\text{ABC}}_0I_t^\nu \mathbf{X}(t) = \frac{1-\nu}{\text{ABC}(\nu)} \mathbf{X}(t) + \frac{\nu}{\text{ABC}(\nu)\Gamma(\nu)} \int_0^t \mathbf{X}(y)(t-y)^{\nu-1} dy. \quad (5)$$

Definition 3 [15] Let a function $\mathbf{X}(t)$, the Caputo derivative can be defined as:

$${}_0^C D_t^\nu \mathbf{X}(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \mathbf{X}'(y)(t-y)^{\nu-1} dy.$$

Definition 4 [21] Assume that $\mathbf{X}(t)$ is a function that is differentiable in a piecewise manner. In this case, one can compute the piecewise derivative of $\mathbf{X}(t)$ using Caputo and ABC operator.

$${}^{\text{PCABC}}_0 D_t^\nu \mathbf{X}(t) = \begin{cases} {}_0^C D_t^\nu \mathbf{X}(t), & 0 < t \leq t_1, \\ {}^{\text{ABC}}_0 D_t^\nu \mathbf{X}(t) & t_1 < t \leq T. \end{cases}$$

The piecewise differential operator ${}^{\text{PCABC}}_0 D_t^\nu$ can be used to calculate the piecewise derivative of $\mathbf{X}(t)$. Specifically, the Caputo operator is employed in the interval $0 < t \leq t_1$, while the ABC operator is used in the interval $t_1 < t \leq T$.

Definition 5 [21] Let $\mathbf{X}(t)$ be a function that is piecewise integrable. We can then compute its piecewise derivative using the Caputo and ABC operators

$${}^{\text{PCABC}}_0 I_t \mathbf{X}(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_{t_1}^t \mathbf{X}(y)(t-y)^{\nu-1} d(y), & 0 < t \leq t_1, \\ \frac{1-\nu}{\text{ABC}\nu} \mathbf{X}(t) + \frac{\nu}{\text{ABC}\nu\Gamma\nu} \int_{t_1}^t \mathbf{X}(y)(t-y)^{\nu-1} d(y) & t_1 < t \leq T. \end{cases}$$

We can represent the piecewise integral operator as ${}^{\text{PCABC}}_0 I_t^\nu$, where the Caputo operator is applied in the interval $0 < t \leq t_1$, and the ABC operator is applied in the interval $t_1 < t \leq T$.

3 Theoretical analysis

This section focuses on establishing the existence results and uniqueness of solution for the proposed system in the context of piecewise functions. We will investigate whether a solution exists for the hypothetical piecewise differentiable function and its particular solution attribute.

For the required solution, we will utilize the model (3) and provide additional clarification as follows:

$$\begin{aligned} {}_0^{\text{PCABC}}\mathbf{D}_t^v \mathcal{U}(t) &= \mathbf{G}(t, \mathcal{U}(t)), \quad 0 < v \leq 1, \\ \mathcal{U}(0) &= \mathcal{U}_0 \end{aligned}$$

is equal to

$$\mathcal{U}(t) = \begin{cases} \mathcal{U}_0 + \frac{1}{\Gamma(v)} \int_0^t \mathbf{G}(y, \mathcal{U}(y))(t-y)^{v-1} dy, & 0 < t \leq t_1, \\ \mathcal{U}(t_1) + \frac{1-v}{\text{ABC}(v)} \mathbf{G}(t, \mathcal{U}(t)) + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) d(y), & t_1 < t \leq T, \end{cases} \quad (6)$$

where

$$\mathcal{U}(t) = \begin{cases} \mathbf{S}(t) \\ \mathbf{I}(t) \\ \mathbf{Y}(t) \\ \mathbf{Z}(t) \end{cases}, \quad \mathcal{U}_0 = \begin{cases} \mathbf{S}_0 \\ \mathbf{I}_0 \\ \mathbf{Y}_0 \\ \mathbf{Z}_0 \end{cases}, \quad \mathcal{U}_{t_1} = \begin{cases} \mathbf{S}_{t_1} \\ \mathbf{I}_{t_1} \\ \mathbf{Y}_{t_1} \\ \mathbf{Z}_{t_1} \end{cases}, \quad \mathbf{G}(t, \mathcal{U}(t)) = \begin{cases} \mathbf{f}_i = \begin{cases} {}^{\text{C}}\mathbf{f}_i(\mathbf{S}, \mathbf{I}, \mathbf{Y}, \mathbf{Z}, t) \\ {}^{\text{ABC}}\mathbf{f}_i(\mathbf{S}, \mathbf{I}, \mathbf{Y}, \mathbf{Z}, t) \end{cases} \end{cases}, \quad (7)$$

we consider $i = 1, 2, 3, 4$ and take $0 < t \leq t < \infty$. Let E_1 represent the space for all piecewise differential function from $C[0, t]$ to R , obviously it is a complete normed space and thus the Banach space $E_1 = C[0, t]$, equipped with a norm

$$\|\mathcal{U}\| = \max_{t \in [0, t]} |\mathcal{U}(t)|.$$

We assume the following growth condition:

(C1) $\exists \mathcal{L}_{\mathbf{G}} > 0; \forall \mathbf{G}, \bar{\mathcal{U}} \in E$ we have

$$|\mathbf{G}(t, \mathcal{U}) - \mathbf{G}(t, \bar{\mathcal{U}})| \leq \mathcal{L}_{\mathbf{G}} |\mathcal{U} - \bar{\mathcal{U}}|,$$

(C2) $\exists C_{\mathbf{G}} > 0$ & $M_{\mathbf{G}} > 0;$

$$|\mathbf{G}(t, \mathcal{U}(t))| \leq C_{\mathbf{G}} |\mathcal{U}| + M_{\mathbf{G}}.$$

If \mathbf{G} is piece-wise continuous on $(0, t_1]$ and $[t_1, T]$ on $[0, \mathcal{T}]$, also satisfying the assumption (C2), then (3) has ≥ 1 solution.

Proof 1 We can apply the Schauder theorem to define a closed subset \mathbf{B} and E in both subintervals of the interval $[0, \mathcal{T}]$.

$$B = \{\mathcal{U} \in E : \|\mathcal{U}\| \leq R_{1,2}, R_{1,2} > 0\},$$

Consider a mapping $\$: \mathbf{B} \rightarrow \mathbf{B}$ and using (6) as

$$\$(\mathcal{U}) = \begin{cases} \mathcal{U}_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} \mathbf{G}(y, \mathcal{U}(y))(t-y)^{v-1} dy, & 0 < t \leq t_1, \\ \mathcal{U}(t_1) + \frac{1-v}{\text{ABC}(v)} \mathbf{G}(t, \mathcal{U}(t)) + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) d(y), & t_1 < t \leq T. \end{cases} \quad (8)$$

Any $\mathcal{U} \in B$, we have

$$\begin{aligned} | \$(\mathcal{U})(t) | &\leq \begin{cases} |\mathcal{U}_0| + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} |\mathbf{G}(y, \mathcal{U}(y))| dy, \\ |\mathcal{U}(t_1)| + \frac{1-v}{\text{ABC}(v)} |\mathbf{G}(t, \mathcal{U}(t))| + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} |\mathbf{G}(y, \mathcal{U}(y))| d(y), \end{cases} \\ &\leq \begin{cases} |\mathcal{U}_0| + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} [C_G |\mathcal{U}| + M_G] dv, \\ |\mathcal{U}(t_1)| + \frac{1-v}{\text{ABC}(v)} [C_G |\mathcal{U}| + M_G] + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} [C_G |\mathcal{U}| + M_G] dy, \end{cases} \\ &\leq \begin{cases} |\mathcal{U}_0| + \frac{\mathbf{T}^v}{\Gamma(v+1)} [C_H |\mathcal{U}| + M_G] = R_1, & 0 < t \leq t_1, \\ |\mathcal{U}(t_1)| + \frac{1-v}{\text{ABC}(v)} [C_G |\mathcal{U}| + M_G] + \frac{v(T-\mathbf{T})^v}{\text{ABC}(v)\Gamma(v)+1} [C_G |\mathcal{U}| + M_G] d(v) = R_2, & t_1 < t \leq T, \end{cases} \\ &\leq \begin{cases} R_1, & 0 < t \leq t_1, \\ R_2, & t_1 < t \leq T. \end{cases} \end{aligned}$$

Upon analyzing the previous equation, it can be concluded that \mathcal{U} belongs to the set \mathbf{B} . As a result, it follows that $\$(\mathbf{B}) \subseteq \mathbf{B}$, indicating the closure and completeness of $\$$. To further showcase its complete continuity, let us consider the initial interval in the Caputo sense as $t_i < t_j \in [0, t_1]$.

$$\begin{aligned} | \$(\mathcal{U})(t_j) - \$(\mathcal{U})(t_i) | &= \left| \frac{1}{\Gamma(v)} \int_0^{t_j} (t_j-y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) dy - \frac{1}{\Gamma(v)} \int_0^{t_i} (t_i-y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) dy \right|, \\ &\leq \frac{1}{\Gamma(v)} \int_0^{t_i} [(t_i-y)^{v-1} - (t_j-y)^{v-1}] |\mathbf{G}(y, \mathcal{U}(y))| dy \\ &\quad + \frac{1}{\Gamma(v)} \int_{t_i}^{t_j} (t_j-y)^{v-1} |\mathbf{G}(y, \mathcal{U}(y))| dy, \\ &\leq \frac{1}{\Gamma(v)} \left[\int_0^{t_i} [(t_i-y)^{v-1} - (t_j-y)^{v-1}] dy + \int_{t_i}^{t_j} (t_j-y)^{v-1} dy \right] (C_H |\mathcal{U}| + M_G), \\ &\leq \frac{(C_G \mathcal{U} + M_G)}{\Gamma(v+1)} [t_j^y - t_i^v + 2(t_j - t_i)^v]. \end{aligned} \quad (9)$$

Next (9), we obtain $t_i \rightarrow t_j$, and then

$$| \$(\mathcal{U})(t_j) - \$(\mathcal{U})(t_i) | \rightarrow 0, \text{ as } t_i \rightarrow t_j.$$

So, \mathcal{S} is equi-continuous in $[0, t_1]$. Consider $t_i, t_j \in [t_1, T]$ in the sense of ABC as

$$\begin{aligned}
 | \mathcal{S}(\mathcal{U})(t_j) - \mathcal{S}(\mathcal{U})(t_i) | &= \left| \frac{1-v}{\text{ABC}(v)} \mathbf{G}(t, \mathcal{U}(t)) + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^{t_j} (t_j - y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) dy, \right. \\
 &\quad \left. - \frac{1-v}{\text{ABC}(v)} \mathbf{G}(t, \mathcal{U}(t)) + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^{t_i} (t_i - y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) dy \right|, \\
 &\leq \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_1}^{t_i} [(t_i - y)^{v-1} - (t_j - y)^{v-1}] |\mathbf{G}(y, \mathcal{U}(y))| dy \\
 &\quad + \frac{v}{\text{ABC}(v)\Gamma(v)} \int_{t_i}^{t_j} (t_j - y)^{v-1} |\mathbf{G}(y, \mathcal{U}(y))| dy, \\
 &\leq \frac{v}{\text{ABC}(v)\Gamma(v)} \left[\int_{t_1}^{t_i} [(t_i - y)^{v-1} - (t_j - y)^{v-1}] dy \right. \\
 &\quad \left. + \int_{t_i}^{t_j} (t_j - y)^{v-1} dy \right] (C_{\mathbf{G}} |\mathcal{U}| + M_{\mathbf{G}}), \\
 &\leq \frac{v(C_{\mathbf{G}} \mathcal{U} + M_{\mathbf{G}})}{\text{ABC}(v)\Gamma(v+1)} [t_j^v - t_i^v + 2(t_j - t_i)^v]. \tag{10}
 \end{aligned}$$

If $t_i \rightarrow t_j$, then

$$| \mathcal{S}(\mathcal{U})(t_j) - \mathcal{S}(\mathcal{U})(t_i) | \rightarrow 0, \text{ as } t_i \rightarrow t_j.$$

Therefore, it can be concluded that the operator \mathcal{S} exhibits equi-continuity in the interval $[t_1, T]$, thus making it an equi-continuous map. By applying the Arzelà-Ascoli theorem, it can be inferred that \mathcal{S} is uniformly continuous, continuous, and bounded. Furthermore, based on the Schauder theorem, it can be established that problem (3) has at least one solution in the subintervals.

Moreover, if the operator \mathcal{S} satisfies the condition of being a contraction mapping with assumption (C1), then the proposed system possesses a unique solution. The mapping $\mathcal{S} : \mathbf{B} \rightarrow \mathbf{B}$ is characterized by being piece-wise continuous, let us consider two elements \mathcal{U} and $\bar{\mathcal{U}} \in B$ on the interval $[0, t_1]$ in the sense of Caputo, as follows:

$$\begin{aligned}
 \| \mathcal{S}(\mathcal{U}) - \mathcal{S}(\bar{\mathcal{U}}) \| &= \max_{t \in [0, t_1]} \left| \frac{1}{\Gamma(v)} \int_0^t (t-y)^{v-1} \mathbf{G}(y, \mathcal{U}(y)) dy - \frac{1}{\Gamma(v)} \int_0^t (t-y)^{v-1} \mathbf{G}(y, \bar{\mathcal{U}}(y)) dy \right|, \\
 &\leq \frac{\mathbf{T}^v}{\Gamma(v+1)} L_{\mathbf{G}} \| \mathcal{U} - \bar{\mathcal{U}} \|. \tag{11}
 \end{aligned}$$

From (11), we have

$$\| \mathcal{S}(\mathcal{U}) - \mathcal{S}(\bar{\mathcal{U}}) \| \leq \frac{\mathbf{T}^v}{\Gamma(v+1)} L_{\mathbf{G}} \| \mathcal{U} - \bar{\mathcal{U}} \|. \tag{12}$$

Consequently, it follows that \mathcal{S} satisfies the contraction mapping condition. Thus, according to the Banach fixed-point theorem, the problem at hand possesses a unique solution in the given subinterval. Additionally, it is worth noting that $t \in [t_1, T]$ in the ABC sense:

$$\| \mathcal{S}(\mathcal{U}) - \mathcal{S}(\bar{\mathcal{U}}) \| \leq \frac{1-v}{\text{ABC}(v)} L_{\mathbf{G}} \| \mathcal{U} - \bar{\mathcal{U}} \| + \frac{v(\mathbf{T} - T^v)}{\text{ABC}(v)\Gamma(v+1)} L_{\mathbf{f}} \| \mathcal{U} - \bar{\mathcal{U}} \|, \tag{13}$$

or

$$\| \$(\mathcal{U}) - \$(\mathcal{U}^{\bar{}})\| \leq L_G \left[\frac{1 - v}{ABC(v)} + \frac{v(T - \mathbf{T})^v}{ABC(v)\Gamma(v + 1)} \right] \|\mathcal{U} - \mathcal{U}^{\bar{}}\|. \tag{14}$$

This implies that \$ satisfies the contraction mapping property. Consequently, the problem being considered has a unique solution in the given sub-interval by virtue of the Banach fixed-point theorem. Therefore, taking into account equations (12) and (14), it can be concluded that the proposed problem has a unique solution on each sub-interval.

Analysis of stability

To prove the Ulam-Hyers stability for the considered model, we need to show that small perturbations in the initial conditions or the parameters of the model result in small perturbations in the solution of the model. This can be done by showing that the operator \$ is Lipschitz continuous with respect to the initial conditions or the parameters of the model.

Definition 6 The considered system (1) is said to be U-H stable if, for every $\aleph > 0$, the inequality holds true.

$$\left| {}^{PCABC}D_t^v \mathbf{U}(t) - \mathbf{f}(t, \mathbf{U}(t)) \right| < \aleph, \text{ for all, } t \in \mathcal{T}, \tag{15}$$

There exists a unique solution $\bar{\mathbf{U}} \in Z$ that is constant $\mathcal{A} > 0$,

$$\|\mathbf{U} - \bar{\mathbf{U}}\|_Z \leq \mathcal{A}\aleph, \text{ for all, } t \in \mathcal{T}, \tag{16}$$

Furthermore, if we consider an increasing function $\Psi : [0, \infty) \rightarrow R^+$, the inequality described above can be expressed as follows:

$$\|\mathbf{U} - \bar{\mathbf{U}}\|_Z \leq \mathcal{A}\Psi(\aleph), \text{ for each, } t \in \mathcal{T},$$

If $\Psi(0) = 0$, then the resulting solution is considered to be generalized U-H (G-H-U) stable.

Remark 1. Assuming that a function $\Psi \in C(\mathcal{T})$ does not depend on $\mathbf{U} \in \mathcal{W}$ and satisfies $\Psi(0) = 0$, we can conclude that:

$$\begin{aligned} |\Psi(t)| &\leq \aleph, t \in \mathcal{T} \\ {}^{PCABC}D_t^v \mathbf{U}(t) &= \mathbf{f}(t, \mathbf{U}(t)) + \Psi(t), t \in \mathcal{T}. \end{aligned}$$

Lemma 1 Consider the function

$${}^{PCABC}D_t^v \mathbf{U}(t) = \mathbf{f}(t, \mathbf{U}(t)), \quad 0 < v \leq 1. \tag{17}$$

The solution of (17) is

$$\mathbf{U}(t) = \begin{cases} \mathbf{U}_0 + \frac{1}{\Gamma(v)} \int_0^t \mathbf{f}(y, \mathbf{U}(y))(t-y)^{v-1} dy, & 0 < t \leq t_1 \\ \mathbf{U}(t_1) + \frac{1-v}{ABC(v)} \mathbf{f}(t, \mathbf{U}(t)) + \frac{v}{ABC(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{f}(y, \mathbf{U}(y)) d(y), & t_1 < t \leq T, \end{cases} \quad (18)$$

$$\|F(\mathbf{U}) - F(\bar{\mathbf{U}})\| \leq \begin{cases} \frac{\mathcal{T}_1^v}{\Gamma(v+1)} \aleph, & t \in \mathcal{T}_1 \\ \left[\frac{(1-v)\Gamma(v) + (\mathcal{T}_2^v)}{ABC(v)\Gamma(v)} \right] \aleph = \Lambda \aleph, & t \in \mathcal{T}_2. \end{cases} \quad (19)$$

Theorem 1 The implication of Lemma (1) is that if $\frac{L_f \mathcal{T}^v}{\Gamma(v)} < 1$, the solution to model (2) is H-U stable as well as G-H-U stable.

Proof 2 If $\mathbf{U} \in \mathcal{W}$ is a solution of (2) and $\bar{\mathbf{U}} \in \mathcal{W}$ is also a unique solution of (2), then we can conclude that

Case:1 for $t \in \mathcal{T}$, we have

$$\begin{aligned} \|\mathbf{U} - \bar{\mathbf{U}}\| &= \sup_{t \in \mathcal{T}} \left| \mathbf{U} - \left(\mathbf{U}_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t_1 - y)^{v-1} \mathbf{f}(y, \bar{\mathbf{U}}(y)) dy \right) \right|, \\ &\leq \sup_{t \in \mathcal{T}} \left| \mathbf{U} - \left(\mathbf{U}_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t_1 - y)^{v-1} \mathbf{f}(y, \bar{\mathbf{U}}(y)) dy \right) \right| \\ &+ \sup_{t \in \mathcal{T}} \left| + \frac{1}{\Gamma(v)} \int_0^{t_1} (t_1 - y)^{v-1} \mathbf{f}(y, \mathbf{U}(y)) dy - \frac{1}{\Gamma(v)} \int_0^{t_1} (t_1 - y)^{v-1} \mathbf{f}(y, \bar{\mathbf{U}}(y)) dy \right|, \\ &\leq \frac{\mathcal{T}_\infty^v}{\Gamma(v+1)} \aleph + \frac{L_f \mathcal{T}_\infty}{\Gamma(v+1)} \|\mathbf{U} - \bar{\mathbf{U}}\|. \end{aligned} \quad (20)$$

On more calculation

$$\|\mathbf{U} - \bar{\mathbf{U}}\| \leq \left(\frac{\frac{\mathcal{T}_\infty^v}{\Gamma(v+1)}}{1 - \frac{L_f \mathcal{T}_\infty}{\Gamma(v+1)}} \right) \aleph. \quad (21)$$

Case:2

$$\begin{aligned} \|\mathbf{U} - \bar{\mathbf{U}}\| &\leq \sup_{t \in \mathcal{T}} \left| \mathbf{U} - \left[\mathbf{U}(t_1) + \frac{1-v}{ABC(v)} [\mathbf{f}(t, \mathbf{U}(t))] \right. \right. \\ &+ \left. \left. \frac{v}{ABC(v)\Gamma(v)} \left[\int_{t_1}^t (t-y)^{v-1} \mathbf{f}(y, \bar{\mathbf{U}}(y)) d(y) \right] \right] \right| \\ &+ \sup_{t \in \mathcal{T}} \frac{1-v}{ABC(v)} |\mathbf{f}(t, \mathbf{U}(t)) - \mathbf{f}(t, \bar{\mathbf{U}}(t))| \\ &+ \sup_{t \in \mathcal{T}} \frac{v}{ABC(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} |\mathbf{f}(y, \mathbf{U}(y)) - \mathbf{f}(y, \bar{\mathbf{U}}(y))| ds. \end{aligned}$$

Using $\Lambda = \left[\frac{(1-v)\Gamma(v)+T_2^v}{ABC(v)\Gamma(v)} \right]$ and further calculation we have

$$\|U - \bar{U}\|_{\mathcal{W}} \leq \Lambda \aleph + \Lambda L_f \|U - \bar{U}\|_{\mathcal{W}},$$

or

$$\|U - \bar{U}\|_{\mathcal{W}} \leq \left(\frac{\Lambda}{1 - \frac{\Lambda}{L_f}} \right) \aleph. \tag{22}$$

We use

$$\mathcal{A} = \max \left\{ \left(\frac{\frac{\mathcal{T}_1}{\Gamma(v+1)}}{1 - \frac{L_f \mathcal{T}_1}{\Gamma(v+1)}} \right), \frac{\Lambda}{1 - \frac{\Lambda L_f}{1 - M_f}} \right\}.$$

Now, from Eq. (21) and (22), we have

$$\|U - \bar{U}\|_{\mathcal{W}} \leq \mathcal{A} \aleph, \text{ at every } t \in \mathcal{T}.$$

Thus, we can conclude that the solution to model (2) is H-U stable. Additionally, if we substitute \aleph with $\Psi(\aleph)$ in (23), we get:

$$\|U - \bar{U}\|_{\mathcal{W}} \leq \mathcal{A} \Psi(\aleph), \text{ at each } t \in \mathcal{T}.$$

Therefore, we can infer that the solution to our proposed model (2) is G-H-U stable based on the fact that $\Psi(0) = 0$.

4 Numerical scheme

Here, we will find the numerical scheme for the considered system (2).

$$\begin{cases} {}_0^{PCABC} \mathbf{D}_t^v \mathbf{S}(t) = \mathbf{S} \left[r \left(1 - \frac{\mathbf{S}}{k} \right) - \frac{\beta \mathbf{I}}{1 + \alpha \mathbf{S}} - p_1 \mathbf{Y} \right], \\ {}_0^{PCABC} \mathbf{D}_t^v \mathbf{I}(t) = \mathbf{I} \left(\frac{\beta \mathbf{S}}{1 + \alpha \mathbf{S}} - \omega \right), \\ {}_0^{PCABC} \mathbf{D}_t^v \mathbf{Y}(t) = \mathbf{Y} \left(-\mathbf{Y}_1 + C_1 p_1 \mathbf{S} - p_2 \mathbf{Z} \right), \\ {}_0^{PCABC} \mathbf{D}_t^v \mathbf{Z}(t) = -\mathbf{Y}_2 \mathbf{Z} + C_2 p_2 \mathbf{Y} \mathbf{Z} + \mu. \end{cases} \tag{23}$$

Using the piecewise-integral of Caputo and ABC derivative, we have

$$\begin{aligned}
 S(t) &= \begin{cases} S_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_1(t, S) dy & 0 < t \leq t_1, \\ S(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_1(t, S) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{f}_1(t, S) dy & t_1 < t \leq T, \end{cases} \\
 I(t) &= \begin{cases} I_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_2(t, I) dy & 0 < t \leq t_1, \\ I(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_2(t, I) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{f}_2(t, I) dy & t_1 < t \leq T, \end{cases} \\
 Y(t) &= \begin{cases} Y_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_3(t, Y) dy & 0 < t \leq t_1, \\ Y(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_3(t, Y) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{f}_3(t, Y) dy & t_1 < t \leq T, \end{cases} \\
 Z(t) &= \begin{cases} Z_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_4(t, Z) dy & 0 < t \leq t_1, \\ Z(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_4(t, Z) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^t (t-y)^{v-1} \mathbf{f}_4(t, Z) dy & t_1 < t \leq T. \end{cases}
 \end{aligned} \tag{24}$$

At $t = t_{n+1}$

$$\begin{aligned}
 S(t) &= \begin{cases} S_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_1(t, S) dy & 0 < t \leq t_1, \\ S(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_1(t, S) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^{t_{n+1}} (t-y)^{v-1} \mathbf{f}_1(t, S) dy & t_1 < t \leq T, \end{cases} \\
 I(t) &= \begin{cases} I_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_2(t, I) dy & 0 < t \leq t_1, \\ I(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_2(t, I) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^{t_{n+1}} (t-y)^{v-1} \mathbf{f}_2(t, I) dy & t_1 < t \leq T, \end{cases} \\
 Y(t) &= \begin{cases} Y_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_3(t, Y) dy & 0 < t \leq t_1, \\ Y(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_3(t, Y) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^{t_{n+1}} (t-y)^{v-1} \mathbf{f}_3(t, Y) dy & t_1 < t \leq T, \end{cases} \\
 Z(t) &= \begin{cases} Z_0 + \frac{1}{\Gamma(v)} \int_0^{t_1} (t-y)^{v-1} C \mathbf{f}_4(t, Z) dy & 0 < t \leq t_1, \\ Z(t_1) + \frac{1-v}{AB(v)} \mathbf{f}_4(t, Z) dy + \frac{v}{AB(v)\Gamma(v)} \int_{t_1}^{t_{n+1}} (t-y)^{v-1} \mathbf{f}_4(t, Z) dy & t_1 < t \leq T. \end{cases}
 \end{aligned}$$

Using the Newton polynomials and some calculation, we have

$$S(t_{n+1}) = \begin{cases} S_0 + \left\{ \begin{aligned} & \frac{(\Delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=2}^i \left[\mathbf{f}_1(S^{x-2}, t_{x-2}) \right] \Pi + \frac{(\Delta t)^{v-1}}{\Gamma(v+2)} \sum_{x=2}^i \left[\mathbf{f}_1(S^{x-1}, t_{x-1}) - {}^C \mathbf{f}_1(S^{x-2}, t_{x-2}) \right] \wedge \\ & + \frac{v(\Delta t)^{v-1}}{2\Gamma(v+3)} \sum_{x=2}^i \left[\mathbf{f}_1(S^x, t_x) - 2 {}^C \mathbf{f}_1(S^{x-1}, t_{x-1}) + {}^C \mathbf{f}_1(S^{x-2}, t_{x-2}) \right] \Delta \end{aligned} \right. \\ S(t_1) + \left\{ \begin{aligned} & \frac{1-v}{ABC(v)} {}^{ABC} \mathbf{f}_1(S^n, t_n) + \frac{v}{ABC(v)} \frac{(\delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=i+3}^n \left[\mathbf{f}_1(S^{x-2}, t_{x-2}) \right] \Pi \\ & + \frac{v}{ABC(v)} \frac{(vt)^{v-1}}{\Gamma(v+2)} \sum_{x=i+3}^n \left[\mathbf{f}_1(S^{x-1}, t_{x-1}) + {}^{ABC} \mathbf{f}_1(S^{x-2}, t_{x-2}) \right] \wedge \\ & + \frac{v}{ABC(v)} \frac{v(vt)^{v-1}}{\Gamma(v+3)} \sum_{x=i+3}^n \left[\mathbf{f}_1(S^x, t_x) - 2 {}^{ABC} \mathbf{f}_1(S^{x-1}, t_{x-1}) + {}^{ABC} \mathbf{f}_1(S^{x-2}, t_{x-2}) \right] \Delta, \end{aligned} \right. \end{cases} \tag{25}$$

$$I(t_{n+1}) = \begin{cases} I_0 + \left\{ \begin{aligned} & \left[\frac{(\Delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=2}^i \left[\mathbf{f}_2(I^{x-2}, t_{x-2}) \right] \Pi + \frac{(\Delta t)^{v-1}}{\Gamma(v+2)} \sum_{x=2}^i \left[\mathbf{f}_2(I^{x-1}, t_{x-1}) - {}^C \mathbf{f}_2(I^{x-2}, t_{x-2}) \right] \right] \wedge \\ & + \frac{v(\Delta t)^{v-1}}{2\Gamma(v+3)} \sum_{x=2}^i \left[\mathbf{f}_2(I^x, t_x) - 2 {}^C \mathbf{f}_2(I^{x-1}, t_{x-1}) + {}^C \mathbf{f}_2(I^{x-2}, t_{x-2}) \right] \Delta \end{aligned} \right. \\ I(t_1) + \left\{ \begin{aligned} & \left[\frac{1-v}{ABC(v)} {}^{ABC} \mathbf{f}_2(I^n, t_n) + \frac{v}{ABC(v)} \frac{(\delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_2(I^{x-2}, t_{x-2}) \right] \right] \Pi \\ & + \frac{v}{ABC(v)} \frac{(vt)^{v-1}}{\Gamma(v+2)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_2(I^{x-1}, t_{x-1}) + ABC \mathbf{f}_2(I^{x-2}, t_{x-2}) \right] \right] \wedge \\ & + \frac{v}{ABC(v)} \frac{v(vt)^{v-1}}{\Gamma(v+3)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_2(I^x, t_x) - 2 {}^{ABC} \mathbf{f}_2(I^{x-1}, t_{x-1}) + {}^{ABC} \mathbf{f}_2(I^{x-2}, t_{x-2}) \right] \Delta, \end{aligned} \right. \end{cases} \quad (26)$$

$$Y(t_{n+1}) = \begin{cases} Y_0 + \left\{ \begin{aligned} & \left[\frac{(\Delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=2}^i \left[\mathbf{f}_3(Y^{x-2}, t_{x-2}) \right] \Pi + \frac{(\Delta t)^{v-1}}{\Gamma(v+2)} \sum_{x=2}^i \left[\mathbf{f}_3(Y^{x-1}, t_{x-1}) - {}^C \mathbf{f}_3(Y^{x-2}, t_{x-2}) \right] \right] \wedge \\ & + \frac{v(\Delta t)^{v-1}}{2\Gamma(v+3)} \sum_{x=2}^i \left[\mathbf{f}_3(Y^x, t_x) - 2 {}^C \mathbf{f}_3(Y^{x-1}, t_{x-1}) + {}^C \mathbf{f}_3(Y^{x-2}, t_{x-2}) \right] \Delta \end{aligned} \right. \\ Y(t_1) + \left\{ \begin{aligned} & \left[\frac{1-v}{ABC(v)} {}^{ABC} \mathbf{f}_3(Y^n, t_n) + \frac{v}{ABC(v)} \frac{(\delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_3(Y^{x-2}, t_{x-2}) \right] \right] \Pi \\ & + \frac{v}{ABC(v)} \frac{(vt)^{v-1}}{\Gamma(v+2)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_3(Y^{x-1}, t_{x-1}) + ABC \mathbf{f}_3(Y^{x-2}, t_{x-2}) \right] \right] \wedge \\ & + \frac{v}{ABC(v)} \frac{v(vt)^{v-1}}{\Gamma(v+3)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_3(Y^x, t_x) - 2 {}^{ABC} \mathbf{f}_3(Y^{x-1}, t_{x-1}) + {}^{ABC} \mathbf{f}_3(Y^{x-2}, t_{x-2}) \right] \Delta, \end{aligned} \right. \end{cases} \quad (27)$$

$$Z(t_{n+1}) = \begin{cases} Z_0 + \left\{ \begin{aligned} & \left[\frac{(\Delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=2}^i \left[\mathbf{f}_4(Z^{x-2}, t_{x-2}) \right] \Pi + \frac{(\Delta t)^{v-1}}{\Gamma(v+2)} \sum_{x=2}^i \left[\mathbf{f}_4(Z^{x-1}, t_{x-1}) - {}^C \mathbf{f}_4(Z^{x-2}, t_{x-2}) \right] \right] \wedge \\ & + \frac{v(\Delta t)^{v-1}}{2\Gamma(v+3)} \sum_{x=2}^i \left[\mathbf{f}_4(Z^x, t_x) - 2 {}^C \mathbf{f}_4(Z^{x-1}, t_{x-1}) + {}^C \mathbf{f}_4(Z^{x-2}, t_{x-2}) \right] \Delta \end{aligned} \right. \\ Z(t_1) + \left\{ \begin{aligned} & \left[\frac{1-v}{ABC(v)} {}^{ABC} \mathbf{f}_4(Z^n, t_n) + \frac{v}{ABC(v)} \frac{(\delta t)^{v-1}}{\Gamma(v+1)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_4(Z^{x-2}, t_{x-2}) \right] \right] \Pi \\ & + \frac{v}{ABC(v)} \frac{(vt)^{v-1}}{\Gamma(v+2)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_4(Z^{x-1}, t_{x-1}) + ABC \mathbf{f}_4(Z^{x-2}, t_{x-2}) \right] \right] \wedge \\ & + \frac{v}{ABC(v)} \frac{v(vt)^{v-1}}{\Gamma(v+3)} \sum_{x=i+3}^n \left[{}^{ABC} \mathbf{f}_4(Z^x, t_x) - 2 {}^{ABC} \mathbf{f}_4(Z^{x-1}, t_{x-1}) + {}^{ABC} \mathbf{f}_4(Z^{x-2}, t_{x-2}) \right] \Delta. \end{aligned} \right. \end{cases} \quad (28)$$

Here

$$\Pi = \begin{bmatrix} (1 - \mathbf{X} + m)^v \left(2(-\mathbf{X} + m)^2 + (3v + 10)(-\mathbf{X} + m) + 2v^2 + 9v + 12 \right) \\ -(-\mathbf{X} + m) \left(2(-\mathbf{X} + m)^2 + (5v + 10)(m - \mathbf{X}) + 6v^2 + 18v + 12 \right) \end{bmatrix},$$

$$\Delta = \begin{bmatrix} (1 - \mathbf{X} + m)^v (3 + n + 2v - \mathbf{X}) \\ -(-\mathbf{X} + m) (m + 3v - \mathbf{X} + 3) \end{bmatrix},$$

$$\Delta = [(1 - \mathbf{X} + m)^v - (-\mathbf{X} + m)^v],$$

and

$$\begin{aligned} {}^C f_1(S, t) &= {}^{ABC} f_1(S, t) = S[r(1 - \frac{S}{k}) - \frac{\beta I}{1+\alpha S} - p_1 Y], \\ {}^C f_3(I, t) &= {}^{ABC} f_3(I, t) = I(\frac{\beta S}{1+\alpha S} - \omega), \\ {}^C f_2(Y, t) &= {}^{ABC} f_2(Y, t) = Y(-Y_1 + C_1 p_1 S - p_2 Z), \\ {}^C f_4(Z, t) &= {}^{ABC} f_4(Z, t) = -Y_2 Z + C_2 p_2 YZ + \mu. \end{aligned}$$

The above Eqs. (25)–(28) are the required solution for the aforementioned system.

5 Results and discussion

In this section we provide the numerical simulation of all four quantities of the considered piecewise problem on different fractional orders using the data of Table 2 taken from [14]. We also check the dynamics on two different intervals in the sense of Caputo and ABC fractional operators. The biological interpretation and their explanation are given in the article [14]. Further, the graphical results are simulated on different values lying between 0 and 1 which shows the total density of each quantity in the form of a continuous spectrum. Furthermore, chaotic dynamics are present in both divergent and convergent cases. All four quantities in the ecosystem oscillate which shows the dependence of one quantity over another one. The piecewise simulation is also provided on two subintervals the total time showing the sudden change dynamics or the crossover behavior

Table 2. Parameters and their description in model (1)

Notations	Numerical values - I	Numerical Values - II
r	4.5	3.8
k	5	15
β	2	2
α	0.25	0.5
ω	1.6	1.6
d_1	1	1
c_1, c_2	1, 1	1, 1
μ	1	1.24143
p_1, p_2	2, 1	2, 1

The graphical representation of the four compartments is given in Figures 1a-1d with chaotic behaviors or phase portraits are given in Figures 1e-1f on different fractional orders using data of Table 2 numerical values-I. The extra information is provided by the fractional operators by taking extra orders lying between 0 and 1. For such data, the susceptible as well as infected plants are fluctuated or oscillated along with the other two quantities of Herbivores having the same dynamic. For this data, the said analysis is not converging faster as shown in the phase portrait figures.

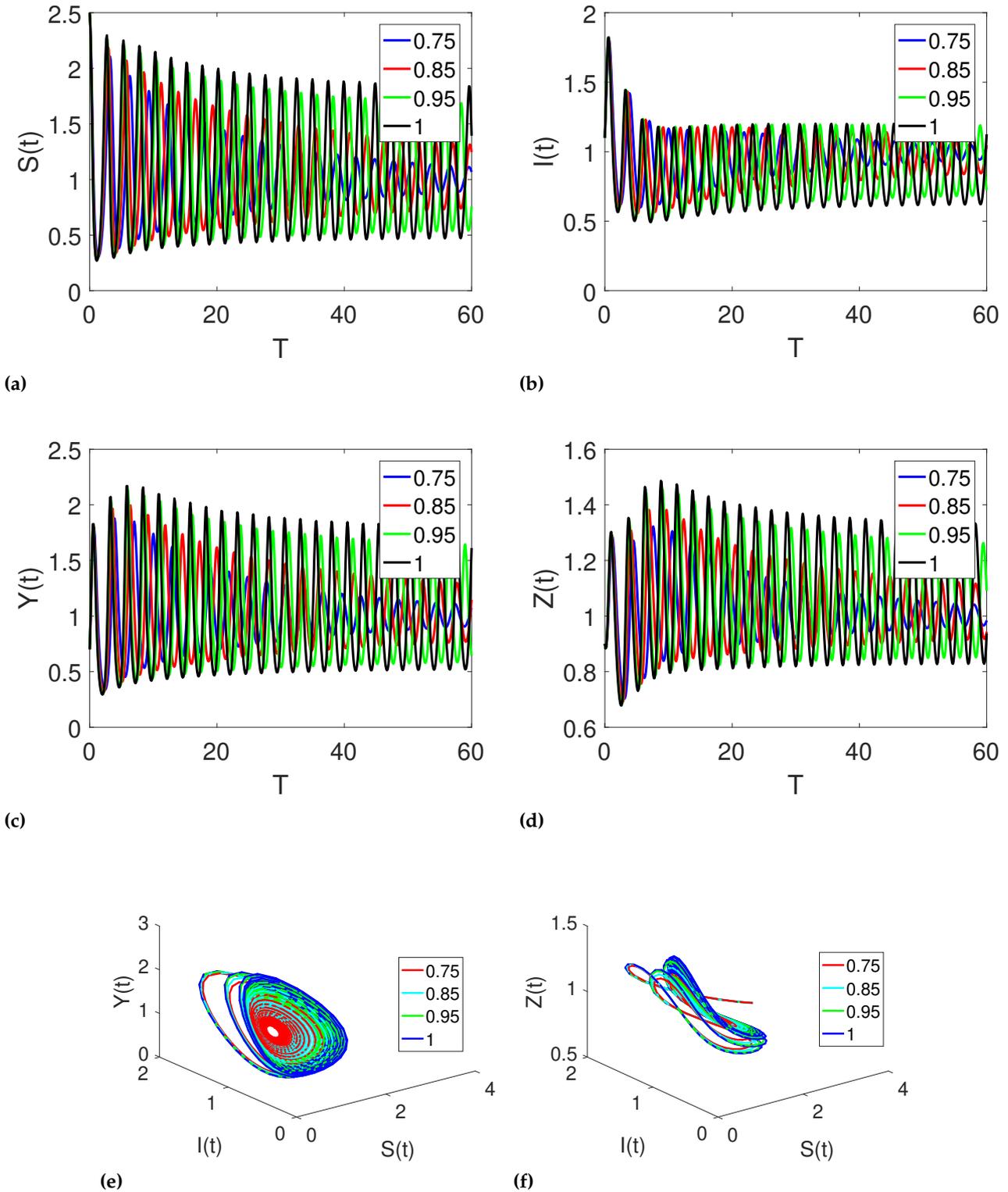


Figure 1. Dynamics of all the four compartments, on different arbitrary fractional orders $\nu = 0.75, 0.85, 0.95, 1$ and time durations on any of the interval for the numerical value-I.

In figures 2a-2f, the graphical representation of the four compartments are given with chaotic behaviors or phase portrait are given on different fractional orders using data of table 2 numerical values-II. The extra information are provided by the fractional derivatives by taking extra orders lies between 0 and 1. For such data the susceptible as well as infected plants are fluctuated or

oscillating along with other two quantities of Herbivores having the same dynamics. But this time the said analysis is converging faster as shown in the phase portrait figures.

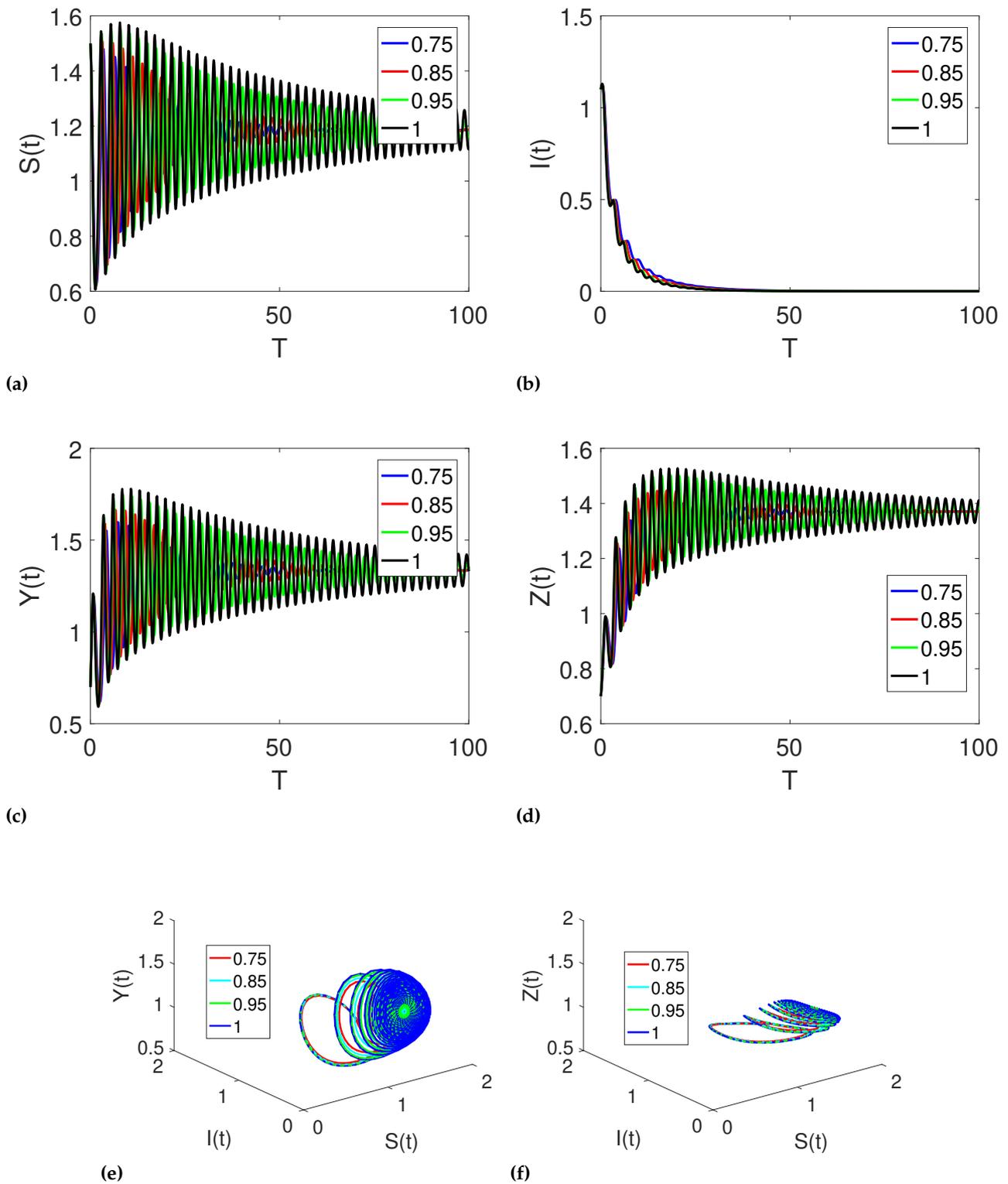


Figure 2. Dynamics of all the four compartments, on different arbitrary fractional orders $\nu = 0.75, 0.85, 0.95, 1$ and time durations on any of the interval for the numerical value-II.

In Figures 3a-3f, the graphical representation of the four compartments are given with chaotic

behaviors or phase portrait are given on different fractional orders using data of table 2 numerical values-I on two sub intervals showing the crossover behaviors. The extra information are provided by the fractional derivatives by taking extra orders lies between 0 and 1. For such data the susceptible as well as infected plants are fluctuated or oscillating along with other two quantities of Herbivores having the oscillating dynamics. But this time the said analysis is not converging faster as shown in the phase portrait figures.

In figures 4a-4d, the graphical representation of the four compartments are given with chaotic behaviors or phase portrait are given on different fractional orders using data of table 2 numerical values-I on two sub intervals showing the crossover behaviors. The more information are provided by the fractional derivatives by taking extra orders lies between 0 and 1. For such data the susceptible as well as infected plants are shifting in one another oscillating along with other two quantities of Herbivores having the overlapping dynamics.

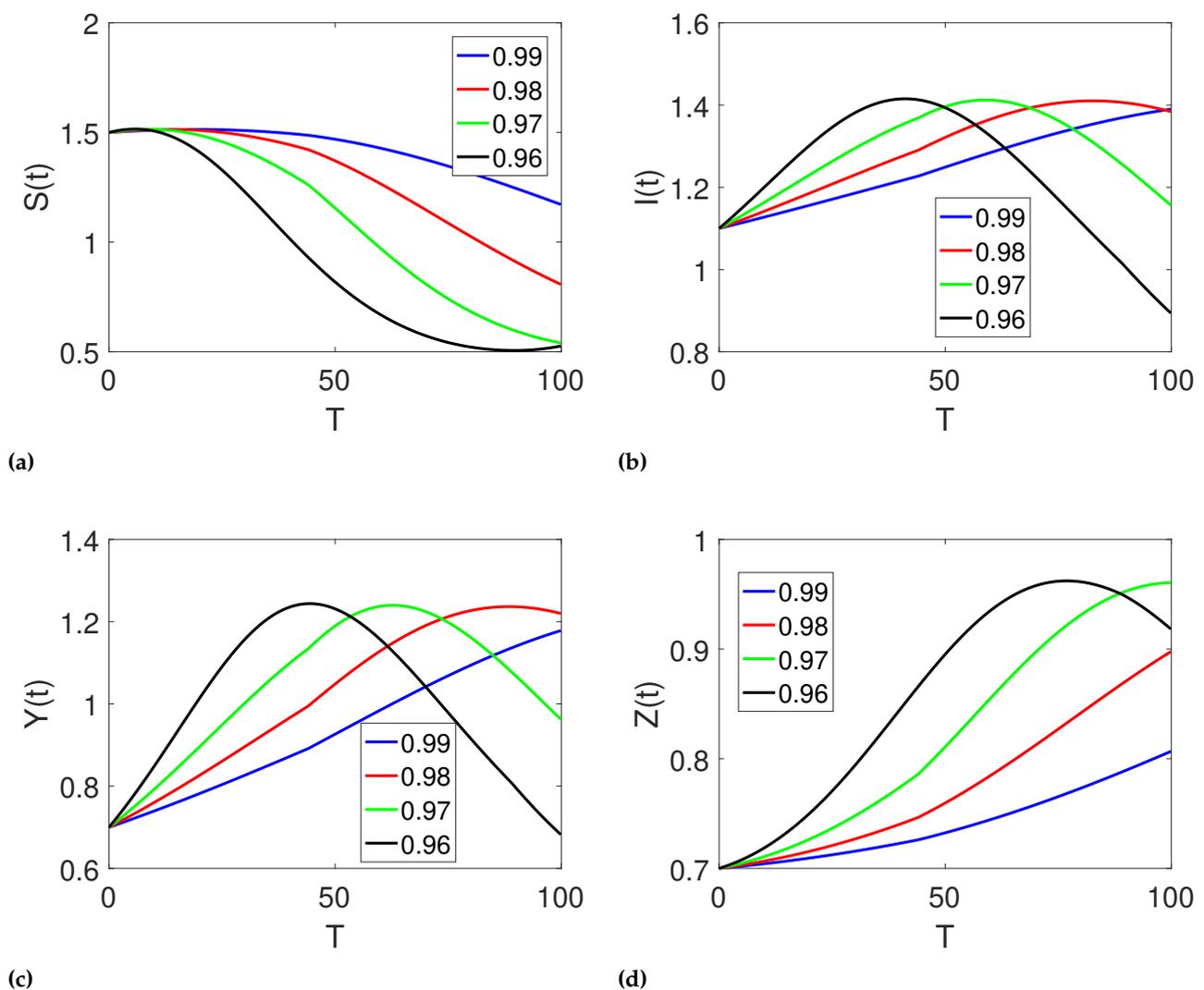


Figure 4. Dynamics of all the four compartments, on different arbitrary fractional orders $\nu = 0.99, 0.98, 0.97, 0.96$ and time durations on any of two sub intervals $[0, t_1], [t_1, t]$ for the numerical value-I showing the crossover dynamics.

In figures 5a-5f, the graphical representation of the four compartments are given with one another depending behaviors or phase portrait are given on different fractional orders using data of table 2

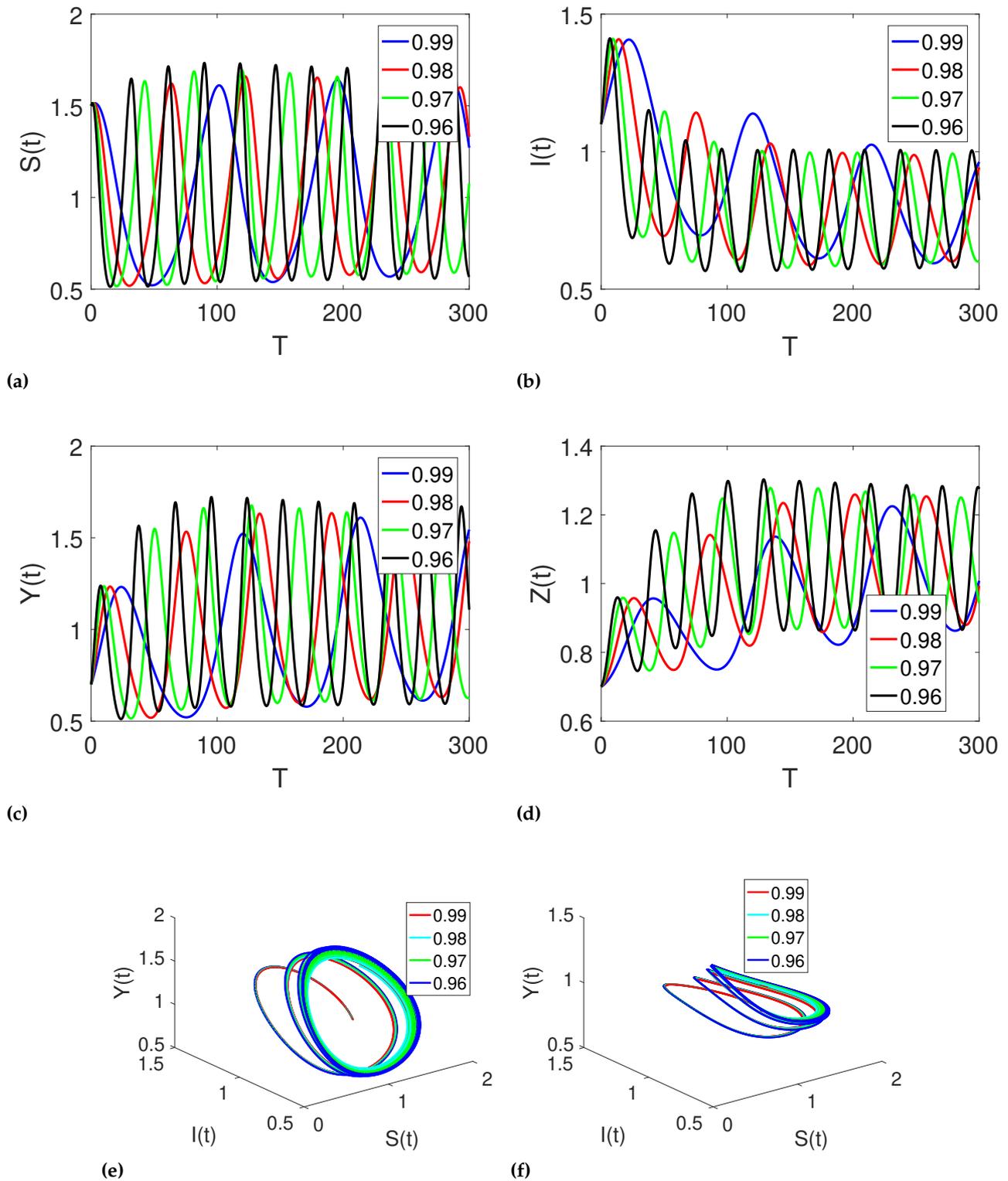


Figure 3. Dynamics of all the four compartments, on different arbitrary fractional orders $\nu = 0.99, 0.98, 0.97, 0.96$ and time durations on any of two sub intervals $[0, t_1], [t_1, t]$ for the numerical value-I.

numerical values-II on two sub intervals showing the crossover behaviors. The extra information are provided by the fractional derivatives by taking extra orders lies between 0 and 1. For such data the susceptible as well as infected plants are fluctuated or oscillating along with other two quantities of Herbivores having the oscillating dynamics which converges faster as shown in the

phase portrait figures.

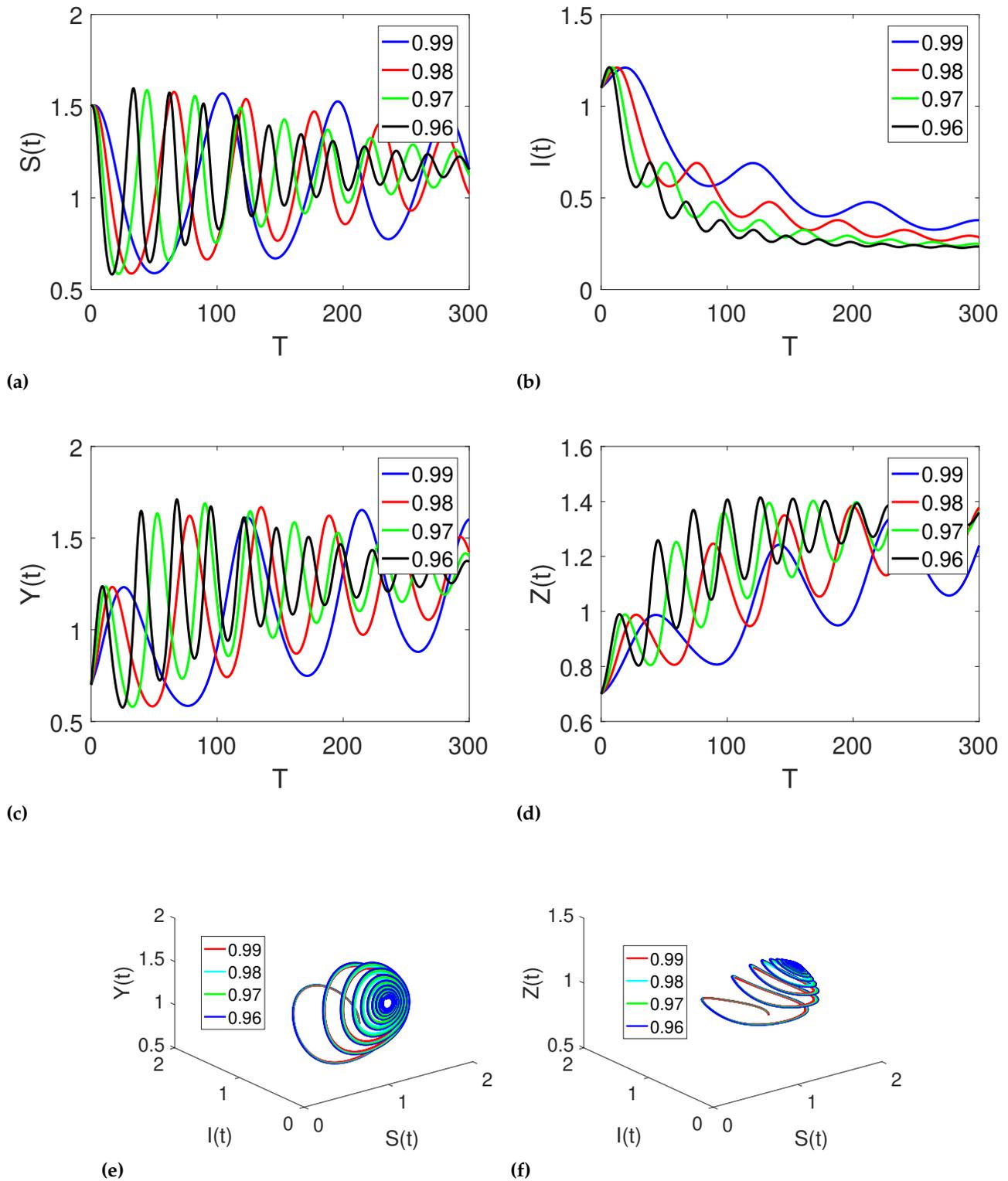


Figure 5. Dynamics of all the four compartments, on different arbitrary fractional orders $\nu = 0.99, 0.98, 0.97, 0.96$ and time durations on any of two sub intervals $[0, t_1], [t_1, t]$ for the numerical value-II.

6 Conclusion

Fractional-order differential equations with short memory play a crucial role in describing various real-world problems. Based on this premise, the current work explores a four-compartmental fractional-order plant model using the concept of piecewise derivative in both the Caputo and ABC sense. The study has further successfully investigated for the existence results, uniqueness of solution, and stability analysis of the considered problem, utilizing the fixed point concept and nonlinear functional analysis tools. Substantial results were obtained and presented in this work. The numerical solutions of the piecewise fractional model were computed using the Newton Polynomial technique in this study. MATLAB-18 was utilized to depict the numerical results for various fractional orders and time durations in this study. The results indicated that the piecewise data provided additional information that described crossover dynamics for different fractional orders. The graphical results obtained from both the piecewise and fractional order analysis were found to be highly intriguing as such analysis are more informative and generalized in the contrast of other relative work. The convergence and stability results are obtained by using fractional, piece wise fractional aspects. Fractional operators are generalized because they have an extra degree of freedom and choices. Therefore, we checked successfully the dynamics of different fractional orders lying between 0 and 1, and compare them with the integer order. On small fractional orders, stability is achieved quickly. Further, the piecewise fractional model is also tested validly for the existence and uniqueness of the solution in the sense of fractional Caputo and Atangana Baleanu operators having a kernel of non-singularity in the form of an exponential function. The approximate or semi-analytical solution is obtained by the technique of piecewise fractional Caputo and ABC derivative having the fractional parameter which shows the extra degree of freedom. Taking different fractional orders, we have simulated the obtained scheme for the first four terms. We also compare the fractional dynamics with the integer order dynamics. All the quantities of the proposed problem are converging to their equilibrium points showing spectrum dynamics with the removal of singularity as well as the crossover or abrupt dynamics.

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Conflicts of interest

The authors declare that they have no conflict of interest.

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Author's contributions

M.U.R.: Writing original draft preparation, Methodology, Validation. M.A.: Investigation, Software, Methodology. D.B.: Writing-Reviewing and Editing, Visualization, Supervision. All authors discussed the results and contributed to the final manuscript.

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References

- [1] Schoonhoven, L.M., Van Loon, J.J. and Dicke, M. Insect-plant biology. *Oxford University Press on Demand*, (2005).
- [2] Strange, R.N. and Scott, P.R. Plant disease: a threat to global food security. *Annual Review of Phytopathology*, 43, 83-116, (2005). [[CrossRef](#)]
- [3] Mithofer, A. and Boland, W. Recognition of herbivory-associated molecular patterns. *Plant Physiology*, 146(3), 825-831, (2008). [[CrossRef](#)]
- [4] Dicke, M. and Sabelis, M.W. How plants obtain predatory mites as bodyguards. *Netherlands Journal of Zoology*, 38(2-4), 148-165, (1987).
- [5] Price, P.W., Bouton, C.E., Gross, P., McPherson, B.A., Thompson, J.N. and Weis, A.E. Interactions among three trophic levels: influence of plants on interactions between insect herbivores and natural enemies. *Annual review of Ecology and Systematics*, 11(1), 41-65, (1980). [[CrossRef](#)]
- [6] Takabayashi, J. and Dicke, M. Plant—carnivore mutualism through herbivore-induced carnivore attractants. *Trends in Plant Science*, 1(4), 109-113, (1996). [[CrossRef](#)]
- [7] Dudareva, N., Negre, F., Nagegowda, D.A. and Orlova, I. Plant volatiles: recent advances and future perspectives. *Critical Reviews in Plant Sciences*, 25(5), 417-440, (2006). [[CrossRef](#)]
- [8] Arimura, G.I., Kost, C. and Boland, W. Herbivore-induced, indirect plant defences. *Biochimica et Biophysica Acta (BBA)-Molecular and Cell Biology of Lipids*, 1734(2), 91-111, (2005). [[CrossRef](#)]
- [9] Dicke, M. Evolution of induced indirect defense of plants (Vol. 1999). Princeton, NJ: Princeton University Press, (1999).
- [10] Tumlinson, J.H., Paré, P.W. and Lewis, W.J. Plant production of volatile semiochemicals in response to insect-derived elicitors. In *Novartis Foundation Symposium*, pp. 95-104, Wiley, Chichester, (1999, January).
- [11] Price, P.W., Denno, R.F., Eubanks, M.D., Finke, D.L. and Kaplan, I. *Insect Ecology: Behavior, Populations and Communities*. Cambridge University Press, (2011).
- [12] Liu, Y.H., Li Liu, D., An, M., Fu, Y.L., Zeng, R.S., Luo, S.M., Wu, H. and Pratley, J. Modelling tritrophic interactions mediated by induced defence volatiles. *Ecological Modelling*, 220(23), 3241-3247, (2009). [[CrossRef](#)]
- [13] Fergola, P. and Wang, W. On the influences of defensive volatiles of plants in tritrophic interactions. *Journal of Biological Systems*, 19(02), 345-363, (2011). [[CrossRef](#)]
- [14] Mukherjee, D. Effect of constant immigration in plant–pathogen–herbivore interactions. *Mathematics and Computers in Simulation*, 160, 192-200, (2019). [[CrossRef](#)]
- [15] Podlubny, I. *Fractional Differential Equations, Mathematics in Science and Engineering*. Academic Press, New York, (1999).
- [16] Atangana, A. and Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Science*, 20(2), 763-785, (2016). [[CrossRef](#)]
- [17] Miller, K.S. Derivatives of noninteger order. *Mathematics magazine*, 68(3), 183-192, (1995). [[CrossRef](#)]
- [18] Zhang, L., ur Rahman, M., Arfan, M. and Ali, A. Investigation of mathematical model of

- transmission co-infection TB in HIV community with a non-singular kernel. *Results in Physics*, 28, 104559, (2021). [[CrossRef](#)]
- [19] Du, M., Wang, Z. and Hu, H. Measuring memory with the order of fractional derivative. *Scientific Reports*, 3(1), 3431, (2013). [[CrossRef](#)]
- [20] Wu, G.C., Gu, C.Y., Huang, L.L. and Baleanu, D. Fractional differential equations of variable order: existence results, numerical method and asymptotic stability conditions. *Miskolc Mathematical Notes*, 23(1), 485-493, (2022). [[CrossRef](#)]
- [21] Atangana, A. and Araz, S.I. New concept in calculus: Piecewise differential and integral operators. *Chaos, Solitons & Fractals*, 145, 110638, (2021). [[CrossRef](#)]
- [22] Ansari, K.J., Asma, Ilyas, F., Shah, K., Khan, A. and Abdeljawad, T. On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative. *Waves in Random and Complex Media*, 1-20, (2023). [[CrossRef](#)]
- [23] Heydari, M.H., Razzaghi, M. and Baleanu, D. Orthonormal piecewise Vieta-Lucas functions for the numerical solution of the one-and two-dimensional piecewise fractional Galilei invariant advection-diffusion equations. *Journal of Advanced Research*, (2022). [[CrossRef](#)]
- [24] Patanarapeelert, N., Asma, A., Ali, A., Shah, K., Abdeljawad, T. and Sitthiwirattam, T. Study of a coupled system with anti-periodic boundary conditions under piecewise Caputo-Fabrizio derivative. *Thermal Science*, 27(1), 287-300, (2023). [[CrossRef](#)]
- [25] Sohail, A., Yu, Z., Arif, R., Nutini, A. and Nofal, T.A. Piecewise differentiation of the fractional order CAR-T cells-SARS-2 virus model. *Results in Physics*, 33, 105046, (2022). [[CrossRef](#)]
- [26] Zeb, A., Atangana, A., Khan, Z.A. and Djillali, S. A robust study of a piecewise fractional order COVID-19 mathematical model. *Alexandria Engineering Journal*, 61(7), 5649-5665, (2022). [[CrossRef](#)]
- [27] Atangana, A. and Araz, S.İ. Modeling third waves of Covid-19 spread with piecewise differential and integral operators: Turkey, Spain and Czechia. *Results in Physics*, 29, 104694, (2021). [[CrossRef](#)]
- [28] Cao, Y., Alamri, S., Rajhi, A.A., Anqi, A.E., Riaz, M. B., Elagan, S.K. and Jawa, T.M. A novel piece-wise approach to modeling interactions in a food web model. *Results in Physics*, 31, 104951, (2021). [[CrossRef](#)]
- [29] El-Shorbagy, M.A., ur Rahman, M. and Alyami, M.A. On the analysis of the fractional model of COVID-19 under the piecewise global operators. *Mathematical Biosciences and Engineering*, 20(4), 6134-6173, (2023). [[CrossRef](#)]
- [30] Khan, J., Ur Rahman, M., Riaz, M.B. and Awrejcewicz, J. A Numerical study on the dynamics of dengue disease model with fractional piecewise derivative. *Fractals*, 30(08), 2240260, (2022). [[CrossRef](#)]
- [31] Qu, H., Saifullah, S., Khan, J., Khan, A., Ur Rahman, M. and Zheng, G. Dynamics of leptospirosis disease in context of piecewise classical-global and classical-fractional operators. *Fractals*, 30(08), 2240216, (2022). [[CrossRef](#)]
- [32] Ansari, K.J., Asma, Ilyas, F., Shah, K., Khan, A. and Abdeljawad, T. On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative. *Waves in Random and Complex Media*, 1-20, (2023). [[CrossRef](#)]
- [33] Shah, K., Abdeljawad, T. and Alrabaiah, H. On coupled system of drug therapy via piecewise equations. *Fractals*, 30(08), 2240206, (2022). [[CrossRef](#)]

- [34] Sohail, A., Yu, Z., Arif, R., Nutini, A. and Nofal, T.A. Piecewise differentiation of the fractional order CAR-T cells-SARS-2 virus model. *Results in Physics*, 33, 105046, (2022). [[CrossRef](#)]
- [35] Atangana, A. and Toufik, M. A piecewise heat equation with constant and variable order coefficients: A new approach to capture crossover behaviors in heat diffusion. *AIMS Mathematics*, 7(5), 8374-8389, (2022). [[CrossRef](#)]

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