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RESEARCH PAPER

Bifurcation analysis of a discrete-time prey-predator model

Parvaiz Ahmad Naik [1,*,‡], **Zohreh Eskandari** [2,‡], **Hossein Eskandari Shahkari** [3,‡] **and Kolade M. Owolabi** [4,‡]

[1]Department of Mathematics and Computer Science, Youjiang Medical University for Nationalities, Baise, Guangxi, China, [2]Department of Mathematics, Faculty of Science, Fasa University, Fasa, Iran, [3]Department of Energy Engineering and Physics, Amirkabir University of Technology, Tehran, Iran, [4]Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State, Nigeria

[*] Corresponding Author

[‡] naik.parvaiz@ymun.edu.cn (Parvaiz Ahmad Naik); z-eskandari@fasau.ac.ir (Zohreh Eskandari); es30377@aut.ac.ir (Hossein Eskandari Shahkari); mkowolax@yahoo.com (Kolade M. Owolabi)

Abstract

This paper investigates the importance of studying the dynamics of predator-prey systems and the specific significance of Neimark-Sacker and period-doubling bifurcations in discrete-time prey-predator models. By conducting a numerical bifurcation analysis and examining bifurcation diagrams and phase portraits, we present important results that differentiate our study from others in the field. Firstly, our analysis reveals the occurrence of Neimark-Sacker and period-doubling bifurcations in the model under certain parameter values. These bifurcations lead to the emergence of stable limit cycles characterized by complex and unpredictable dynamics. This finding emphasizes the inherent complexity and nonlinearity of predator-prey systems and contributes to a deeper understanding of their dynamics. Additionally, our study highlights the advantages and limitations of employing discrete-time models in population dynamics research. The use of discrete-time models allows for a more tractable analysis while still capturing significant aspects of ecological systems. In conclusion, this study holds importance in shedding light on the dynamics of predator-prey systems and the specific role of Neimark-Sacker and period-doubling bifurcations. Our findings contribute to the understanding of predator-prey systems and offer implications for ecological management strategies.

Keywords: Predator-prey model; normal form; bifurcation; numerical continuation method

AMS 2020 Classification: 34C23; 37G15; 39A28; 65P30; 92D25

1 Introduction

Predator-prey models have a long and significant history in ecological research. Notably, pioneers in the field, including Alfred J. Lotka and Vito Volterra, made substantial contributions to the development of these models in the early 20^{th} century [1–3]. These models play a crucial role

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in enhancing our understanding of predator-prey interactions by examining the changes in population sizes of both species over time. They provide valuable insights into the dynamics and complexities of these ecological relationships.

In recent years, discrete-time predator-prey models have gained increasing attention as a means to study ecological systems [4, 5]. These models offer various advantages over continuous-time models, including ease of implementation and analysis, as well as the ability to incorporate population fluctuations and stochastic effects. One prominent discrete-time model widely utilized in predator-prey studies is the model, represented by the following equations [6–8]:

$$\begin{cases} x_p(n+1) &= x_p(n)(1 + r(1 - x_p(n)/k) - a \cdot y_p(n)), \\ y_p(n+1) &= y_p(n)(1 - b + c x_p(n)/y_p(n)), \end{cases} \tag{1}$$

where $x_p(n)$ and $y_p(n)$ denote the prey and predator populations at time n, respectively. The parameters r, k, a, b, and c play crucial roles in the model.

Understanding the significance of these variables in the context of predator-prey dynamics is essential. For instance, the intrinsic growth rate (r) determines the prey population's growth rate in the absence of predators, while the carrying capacity (k) represents the maximum population size that the environment can support. The predation rate (a), natural mortality rate (b), and conversion factor (c) influence prey consumption and the predator population's growth rate. By carefully studying and manipulating these variables, researchers can gain insights into the intricate dynamics of predator-prey interactions in nature.

In this paper, our focus lies on conducting a thorough bifurcation analysis of the model mentioned above. Bifurcations are abrupt qualitative changes that occur in the behavior of a dynamic system as a parameter is varied. Understanding the potential bifurcations within the model is crucial for comprehending predator-prey dynamics and developing effective management strategies for ecosystems and natural resources.

To enhance the literature review, we will extensively explore and incorporate relevant papers in this field. We will delve into the historical development of predator-prey models, including the influential Lotka-Volterra model and its various extensions. Furthermore, we will discuss the advantages and limitations of discrete-time models and review the existing literature on bifurcation analysis in predator-prey systems, with particular emphasis on Neimark-Sacker and period-doubling bifurcations. We aim to establish a comprehensive understanding of the current state of research in this area.

Additionally, we will present our numerical bifurcation analysis of the model, systematically varying the model parameters and analyzing the resulting bifurcation diagrams and phase portraits. By identifying the parameter values at which Neimark-Sacker and period-doubling bifurcations occur, we will shed light on their implications for predator-prey dynamics and the development of effective management strategies.

Overall, our paper contributes to a deeper understanding of predator-prey system dynamics and the occurrence of bifurcations. By incorporating a robust literature review, we aim to provide valuable insights into the existing body of research. Our findings have significant implications for the study of ecosystems, population dynamics, and the formulation of effective management strategies for natural resources. The major contributions of this paper include:

i. Thorough Bifurcation Analysis: This paper conducts a comprehensive bifurcation analysis of a discrete-time predator-prey model. By systematically varying the model parameters and analyzing the resulting bifurcation diagrams and phase portraits, the study identifies the parameter values at which Neimark-Sacker and period-doubling bifurcations occur. This analysis provides valuable insights into the dynamics of predator-prey interactions and contributes to a deeper understanding of ecosystem dynamics.

ii. Literature Review: The paper incorporates an extensive literature review, delving into the historical development of predator-prey models, including the influential Lotka-Volterra model and its extensions. It also discusses the advantages and limitations of discrete-time models and reviews existing research on bifurcation analysis in predator-prey systems, with a focus on Neimark-Sacker and period-doubling bifurcations. This comprehensive review enhances the understanding of the current state of research in this field.

iii. Numerical Continuation Techniques: The paper presents numerical continuation techniques for the discrete-time predator-prey model, allowing for the exploration of parameter variations and the analysis of bifurcation phenomena. By employing these techniques, the study provides a practical approach to studying predator-prey dynamics and offers insights into the development of effective management strategies for natural resources.

Overall, the contributions of this paper advance the understanding of predator-prey system dynamics and the occurrence of bifurcations. The combination of a thorough bifurcation analysis, a comprehensive literature review, and the application of numerical continuation techniques establish valuable insights into the intricate dynamics of predator-prey interactions and their implications for ecological management.
Mathematical modeling and bifurcation analysis do not find application only in the field of mathematical ecology such as predator-prey modeling [6–13], but are widely used to derive dynamic process models in the field of mathematical epidemiology such as disease modeling [14–28], artificial intelligence [29–33], and other fields of science and technology [34–41]. Researchers continuously work in the field of mathematical ecology to present the challenges and suggest solutions to help future researchers of the respective field understand the existing research gaps. The subsequent sections of this paper are organized as follows: after the introduction in Section 1, Section 2 provides the details about the model description. Section 3 focuses on the existence and feasibility of fixed points, while Section 4 carries out a bifurcation analysis of the positive fixed point. In Section 5, we present numerical continuation techniques for the discrete-time prey-predator system described by the model. Finally, we conclude our study with remarks in Section 6, summarizing the key findings and suggestions for future research.

2 Model description

The predator-prey model considered in this study is a discrete-time model that builds upon the classic Lotka-Volterra framework. It incorporates key ecological dynamics and assumptions to capture the interactions between prey and predator populations over discrete time steps.
The model describes the population dynamics of two species: the prey (denoted as x_p) and the predator (denoted as y_p). The population sizes of both species are assumed to change over discrete time intervals.
The model formulation assumes the following dynamics:

i. Prey Growth: The prey population experiences logistic growth, influenced by its intrinsic growth rate (r) and the carrying capacity of the environment (k). The logistic growth term ensures that the prey population growth slows down as it approaches the carrying capacity, representing limited resources and environmental constraints.

ii. Predator-Prey Interaction: The model assumes that the predator population is primarily sustained by predation on the prey population. The predation rate (a) determines the impact of predators on prey consumption. As the prey population increases, the predation rate increases, reflecting a functional response.

iii. Predator Dynamics: The predator population is influenced by prey consumption and natural mortality. The conversion factor (c) represents the efficiency of converting consumed prey into predator population growth, while the natural mortality rate (b) accounts for non-predation-related predator mortality.

The discrete-time equations describing the model are as follows:

$$\begin{cases} x_p(n+1) &= x_p(n)\left(1 + r\left(1 - \frac{x_p(n)}{k}\right) - a \cdot y_p(n)\right), \\ y_p(n+1) &= y_p(n)\left(1 - b + \frac{c \cdot x_p(n)}{y_p(n)}\right), \end{cases} \tag{2}$$

where $x_p(n)$ and $y_p(n)$ represent the prey and predator populations at discrete time step n, respectively.

It is important to note that while this model is based on the Lotka-Volterra framework, it may incorporate modifications or refinements to capture specific ecological dynamics or address limitations observed in previous models. However, without further details on the specific modifications, it is challenging to determine the novelty of this particular model.

The model provides a simplified representation of predator-prey dynamics, focusing on the key factors that influence population interactions. By analyzing the model's dynamics, such as bifurcations and stability, we can gain insights into the complex dynamics of predator-prey systems and their implications for ecosystem management and conservation strategies.

In the following sections, we will delve into the analysis of this model, exploring its steady-state solutions, conducting bifurcation analyses, and utilizing numerical techniques to gain a deeper understanding of predator-prey dynamics and their implications for ecological systems.

3 Fixed points: existence and feasibility

The fixed points of system (1) can be obtained from the following equations:

$$\begin{cases} x_p\left(1 + r\left(1 - \frac{x_p}{k}\right) - a\, y_p\right) &= x_p, \\ y_p\left(1 - b + \frac{c x_p}{y_p}\right) &= y_p. \end{cases} \tag{3}$$

Solving for the fixed points of system (1) leads to the identification of the positive fixed point as:

$$\mathcal{P}_* = \left(\frac{rkb}{ack + br}, \frac{crk}{ack + br}\right).$$

In the upcoming section, we will discuss one-parameter bifurcations of the positive fixed point $\mathcal{P}_*$. Understanding the existence and feasibility of fixed points is essential in analyzing the behavior and stability of dynamical systems. By investigating the properties of fixed points, researchers can gain insights into the long-term dynamics of the system and predict how it will evolve under different conditions and perturbations.

4 Bifurcation analysis of the positive fixed point $\mathcal{P}_*$

The map $\mathcal{M}^{pp}(\mathcal{P}, \Omega)$ represents the system (1), where $\mathcal{P} = (x_p, y_p)^T$ and $\Omega = (a, b, c, r, k)^T$. This map can be expanded using the Taylor series expansion, such that:

$$\mathcal{M}^{pp}(\mathcal{P}, \Omega) = \mathcal{J}_1(\mathcal{P}, \Omega)\mathcal{P} + \frac{1}{2!}\mathcal{J}_2(\mathcal{P}, \mathcal{P}) + \frac{1}{3!}\mathcal{J}_3(\mathcal{P}, \mathcal{P}, \mathcal{P}) + \mathcal{O}(\| \mathcal{P} \|^4),$$

where $\| \mathcal{P} \| = \sqrt{x_p^2 + y_p^2}$, and $\mathcal{J}_1$. The elements of $\mathcal{J}_2$ and $\mathcal{J}_3$ are given by:

$$\mathcal{J}_{2,i}(\Gamma, \Sigma) = \sum_{j,k=1}^{2} \frac{\partial^2 \mathcal{M}_i^{pp}(\mathcal{P}, \Omega)}{\partial \mathcal{P} j \partial \mathcal{P} k} \gamma_j \sigma_k,$$

$$\mathcal{J}_{3,i}(\Gamma, \Sigma, \Upsilon) = \sum_{j,k,l=1}^{2} \frac{\partial^3 \mathcal{M}_i^{pp}(\mathcal{P}, \Omega)}{\partial \mathcal{P}_j \partial \mathcal{P}_k \partial \mathcal{P}_l} \gamma_j \sigma_k \upsilon_l,$$

where $\Gamma = (\gamma_1, \gamma_2)^T$, $\Sigma = (\sigma_1, \sigma_2)^T$, and $\Upsilon = (\upsilon_1, \upsilon_2)^T$. By studying the eigenvalues and eigenvectors of the Jacobian matrices, one can determine the stability and bifurcations of the fixed point $\mathcal{P}_*$ under different parameter values. This analysis is crucial in understanding the complex and fascinating dynamics of the predator-prey system.

One parameter bifurcation

This section considers the bifurcation parameter a, which plays a crucial role in determining the behavior and stability of the predator-prey system. By varying the value of a, researchers can explore how the system responds to changes in the predation rate of the predator population on the prey population. This analysis is essential in understanding the dynamics of the system and predicting its long-term behavior under different conditions and scenarios.

Theorem 1 *Under the influence of the bifurcation parameter a, the positive fixed point $\mathcal{P}_*$ undergoes a period-doubling bifurcation at a critical value of a, given by:*

$$a = a_{PD} = -\frac{br\,(br - 2\,b - 2\,r + 4)}{ck\,(br - 2\,b + 4)}.$$

During this bifurcation, the dynamics of the predator-prey system exhibit a doubling of the period of oscillation. This phenomenon is a hallmark of nonlinear dynamical systems and can have significant implications for the system's behavior and stability. Understanding the nature and timing of bifurcations is crucial for predicting the long-term behavior of the system and developing effective control strategies to manage and mitigate potential risks.

Proof When $a = a_{PD}$, the Jacobian matrix of the system evaluated at the fixed point $\mathcal{P}_*$ is given by:

$$\mathcal{J}_1 = \begin{pmatrix} -1 - 1/2\,br + b & 1/2\,\frac{(r-2)(b-2)b}{c} \\ c & 1 - b \end{pmatrix}.$$

The eigenvalues of $\mathcal{J}_1$ at $a = a_{PD}$ are:

$$\lambda_{PD}^1 = -1, \qquad \lambda_{PD}^2 = -1/2\,br + 1.$$

A period-doubling bifurcation occurs on the curve $\mathcal{T}_{PD}^{pp} = \left\{(x_p, y_p, a, b, c, r, k);\, a = a_{PD}\right\}$ if $\lambda_{PD}^2 \neq \pm 1$. In this case, the dynamics of the system exhibit a doubling of the period of oscillation. The map $\mathcal{M}^{pp}$ can be expressed as:

$$\eta_{PD} \mapsto -\eta_{PD} + \frac{1}{6}\widehat{\beta_{PD}^{pp}}\eta_{PD}^3 + \mathcal{O}(\eta_{PD}^4),$$

where $\widehat{\beta^{pp}_{PD}}$ is given by:

$$\widehat{b^{pp}_{PD}} = \frac{1}{6}\langle w_{PD}, \mathcal{J}_3(v_{PD}, v_{PD}, v_{PD}) \\ +3\mathcal{J}_2\left(v_{PD}, (I_2 - \mathcal{J}_1)^{-1}\mathcal{J}_2(v_{PD}, v_{PD})\right)\rangle.$$

Here, $\mathcal{J}_1 v_{PD} = -v_{PD}$ and $\mathcal{J}_1^T w_{PD} = -w_{PD}$, with $\langle w_{PD}, v_{PD}\rangle = 1$. Using these relationships, we find that:

$$v_{PD} = \begin{pmatrix} \frac{b-2}{c} \\ 1 \end{pmatrix}, \quad w_{PD} = \begin{pmatrix} 2\,\frac{c}{br-4} \\ \frac{(r-2)b}{br-4} \end{pmatrix}.$$

Substituting these expressions into the formula for $\widehat{\beta^{pp}_{PD}}$, we obtain:

$$\widehat{\beta^{pp}_{PD}} = 16\,\frac{r\,(b-2)^3\,(r+2)}{(br-2\,b+4)^2\,k^2c^2\,(br-4)}.$$

The sign of $\widehat{\beta^{pp}_{PD}}$ determines the nature of the bifurcation. A stable (unstable) double period cycle occurs when $\widehat{\beta^{pp}_{PD}} > 0$ ($\widehat{\beta^{pp}_{PD}} < 0$), and the bifurcation is supercritical (subcritical).
Therefore, we have shown that when $a = a_{PD}$, the fixed point $\mathcal{P}_*$ undergoes a period-doubling bifurcation, and the nature of the bifurcation depends on the sign of $\widehat{\beta^{pp}_{PD}}$. This completes the proof of Theorem 1. ■
The following theorem characterizes the behavior of the predator-prey system in the presence of a specific value of the bifurcation parameter a:

Theorem 2 *When $a = a_{NS} = -\frac{r(br-b-r)}{ck(r-1)}$, the positive fixed point $\mathcal{P}_*$ undergoes a Neimark-Sacker bifurcation.*

Proof We begin by considering the case where $a = a_{NS}$, where a_{NS} is defined as $a_{NS} = -\frac{r(br-b-r)}{ck(r-1)}$. In this case, the Jacobian matrix of the predator-prey system evaluated at the positive fixed point $\mathcal{P}_*$ takes the form:

$$\mathcal{J}_1 = \begin{pmatrix} -br+b+1 & \frac{b(br-b-r)}{c} \\ c & 1-b \end{pmatrix}.$$

The eigenvalues of $\mathcal{J}_1$ are complex conjugate pairs given by:

$$\lambda^{1,2}_{NS} = e^{\pm i\theta_0} = -\frac{1}{2}br + 1 \pm i\frac{1}{2}\sqrt{-b^2r^2 + 4br},$$

where θ_0 is a real number.
On the curve $\mathcal{T}^{pp}_{NS} = \{(x_p, y_p, a, b, c, r, k); a = a_{NS})\}$, a Neimark-Sacker bifurcation is possible. The map $\mathcal{M}^{pp}$ can be expressed as:

$$\eta_{NS} \mapsto e^{i\theta_0}\eta_{NS} + \widehat{\delta^{pp}_{NS}}\eta^2_{NS}\overline{\eta_{NS}} + \mathcal{O}(|\eta_{NS}|^4),$$

where $\widehat{\delta^{pp}_{NS}}$ is given by:

$$\widehat{\delta^{pp}_{NS}} = \frac{1}{2}\langle w_{NS}, \mathcal{J}_3\left(v_{NS}, v_{NS}, \overline{v_{NS}}\right) + 2\mathcal{J}_2\left(v_{NS}, \left(I_2 - \mathcal{J}_1\right)^{-1}\mathcal{J}_2(v_{NS}, \overline{v_{NS}})\right)$$
$$+\mathcal{J}_2\left(\overline{v_{NS}}, \left(e^{2i\theta_0}I_2 - \mathcal{J}_1\right)^{-1}\mathcal{J}_2(v_{NS}, v_{NS})\right)\rangle,$$

where v_{NS} and w_{NS} are the eigenvectors of $\mathcal{J}_1$ corresponding to the eigenvalues λ^1_{NS} and λ^2_{NS}, respectively. We have also used the notation $\langle w_{NS}, v_{NS}\rangle = 1$. A non-degenerate Neimark-Sacker bifurcation occurs on the curve $\mathcal{T}^{pp}_{NS}$ when $\widehat{\delta^{pp}_{NS}} \neq 0$. The sign of $\widehat{\sigma^{pp}_{NS}} = \Re\left(e^{-i\theta_0}\widehat{\delta^{pp}_{NS}}\right)$ determines the type of bifurcation. Specifically, if $\widehat{\sigma^{pp}_{NS}} < 0$ ($\widehat{\sigma^{pp}_{NS}} > 0$), there is a stable (unstable) closed invariant curve, and the bifurcation is supercritical (subcritical). For further details, see [42–44]. ■

5 Numerical continuation

A numerical verification of analytical findings is very important from a practical viewpoint. This verification is not possible without the help of computer software like MATLAB as well as Mathematica. In this current section, MATCONTM is used for confirming the analytical results and obtaining further information about the behavior of $\mathcal{M}^{pp}$, [45]. It uses numerical continuation as a method to confirm the analytical results. This section will examine two distinct cases, providing a more detailed analysis of the predator-prey system under different conditions.

Case 1: Our assumptions are that $b = 2.5,\ c = 0.05,\ r = 1.2,\ k = 200$ are fixed parameters. One parameter bifurcation is produced by the continuation method by varying the free parameter a:

i) For $a = a_{PD} = 0.060000$, there is a period-doubling bifurcation (PD) at $\mathcal{P}_{PD} = (166.666667, 3.333333)$, with $\widehat{\beta^{pp}_{PD}} = -1.90090 \times 10^{-04}$.

Since $\widehat{\beta^{pp}_{PD}} < 0$, we conclude that the period-doubling bifurcation is subcritical. Figure 1 displays the period-doubling bifurcation diagram.

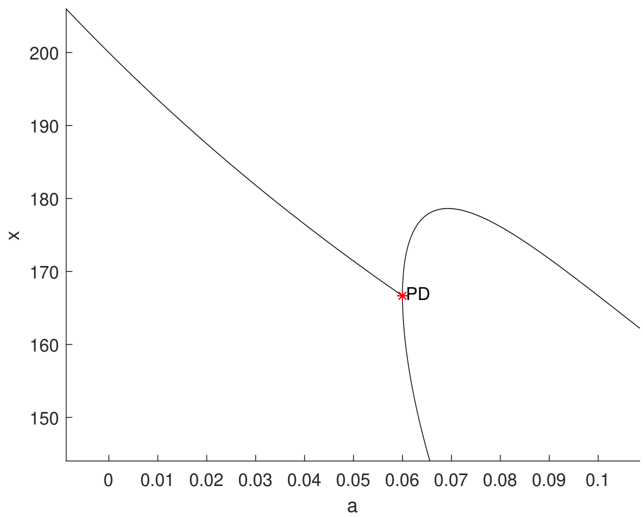

Figure 1. The period-doubling bifurcation diagram

Case 2: Our assumptions are that $b = 0.15$, $c = 0.05$, $r = 1.2$, $k = 200$ are fixed parameters. One parameter bifurcation is produced by the continuation method by varying the free parameter a:

i) For $a = a_{NS} = 0.702000$, there is a Neimark-Sacker bifurcation (NS) at $\mathcal{P}_{NS} = (5.000000, 1.666667)$, with $\widehat{\sigma_{NS}^{pp}} = 8.873596 \times 10^{-06}$.

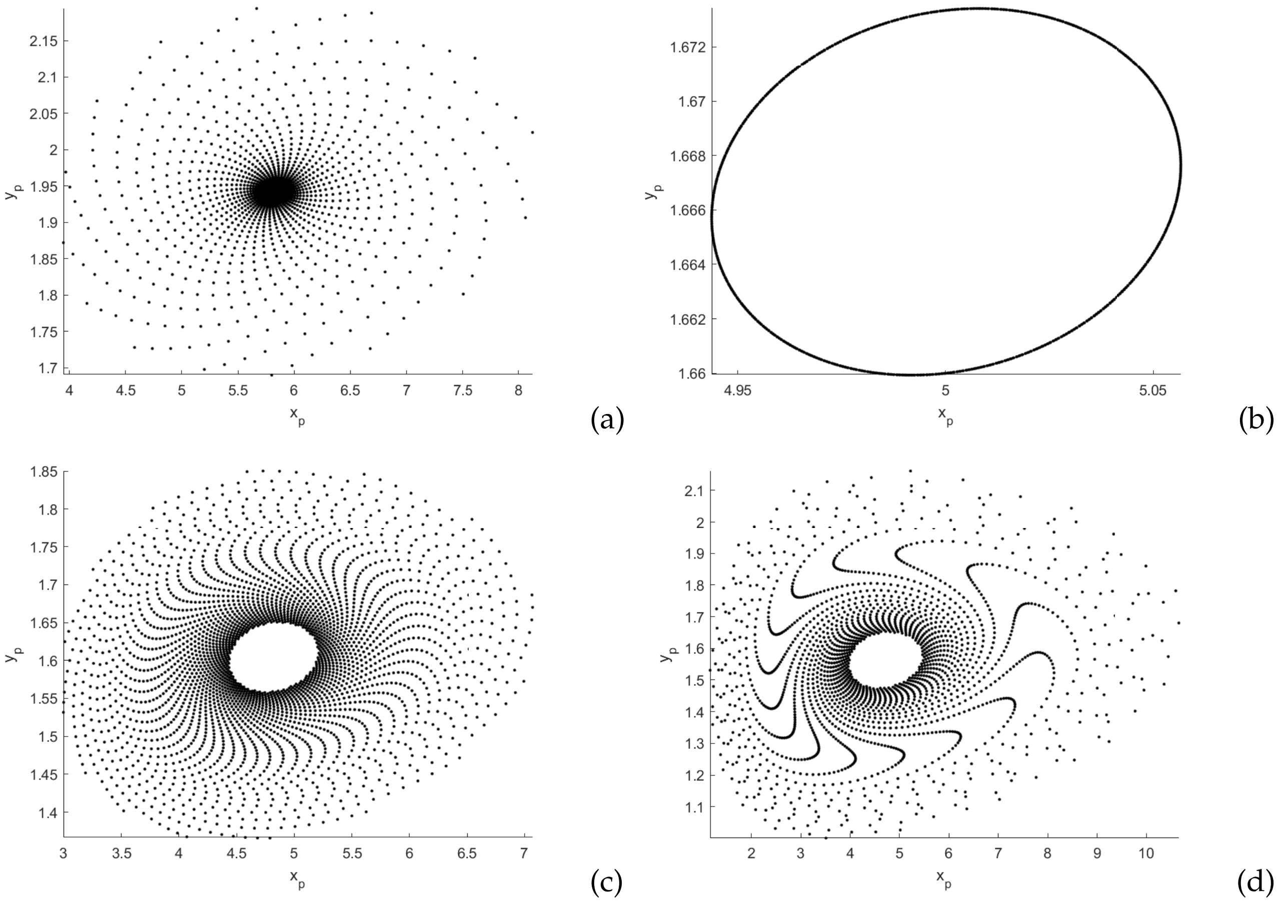

Figure 2. Phase portraits of $\mathcal{M}^{pp}$ near $\mathcal{P}_{NS}$
(a) $a = a_{NS} = 0.6$ (b) $a = a_{NS} = 0.702$ (c) $a = a_{NS} = 0.73$ (d) $a = a_{NS} = 0.75$

Phase portraits of $\mathcal{M}^{pp}$ near $\mathcal{P}_{NS}$ are depicted in Figure 2. Figure 2(a) illustrates the presence of a stable fixed point preceding the Neimark-Sacker point. In Figure 2(b), a closed invariant curve is observed at the Neimark-Sacker point. The closed invariant curve is shown to be broken after the Neimark-Sacker point in Figure 2(c). Finally, Figure 2(d) demonstrates the existence of a chaotic attractor.

6 Conclusion

In this study, we conducted a comprehensive investigation into the Neimark-Sacker and period-doubling bifurcations occurring in a discrete-time prey-predator model using numerical bifurcation analysis. By identifying the specific bifurcation scenarios and their consequences, we gain insights that can inform ecological management and the development of effective strategies for natural resource management. Our analysis revealed that the model displays intricate and unpredictable dynamics under certain parameter values, which can be attributed to the emergence of stable limit cycles. These findings significantly contribute to our understanding of the dynamics

of predator-prey systems and the potential bifurcations that can arise, thereby bearing implications for the study of ecosystems and population dynamics.
Furthermore, we thoroughly discussed the advantages and limitations associated with employing discrete-time models in population dynamics research. Discrete-time models offer notable advantages, including easier implementation and analysis, as well as the ability to account for population fluctuations and stochastic effects. However, it is important to acknowledge their limitations, such as assuming discrete generations and the potential for measurement errors.
Our study emphasizes the significance of comprehending the dynamics of predator-prey systems and the potential bifurcations that can manifest, providing a valuable framework for future research endeavors in this domain. Further investigations could explore the effects of diverse model parameters and initial conditions on the system's dynamics, as well as how the outcomes of our analysis can be effectively applied to real-world ecological systems.
To summarize, our study significantly advances our understanding of the dynamics of predator-prey systems and the potential bifurcations that can arise within discrete-time models. These findings hold crucial implications for the study of ecosystems and population dynamics, thereby aiding the development of effective management strategies for natural resources.

Declarations

Ethical approval

Not applicable.

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no known competing interests regarding the work reported in this article.

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Author's contributions

P.A.N.: Conceptualisation, Analysis, Methodology, Funding acquisition, Supervision, Writing-Original draft. Z.E.: Software, Numerical simulation, Investigation, Visualization, Writing-Original draft. H.E.S.: Numerical simulations, Resources, Validations, K.M.O.: Writing-Reviewing and Editing, Validation. All authors documented the results, prepared the manuscript, and worked on enhancing writing quality. All authors read the final version of the manuscript and approved the results.

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